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Mathematical Analysis

# Closure of the set of pseudodifferential operators

## *Adhérence de l'ensemble des opérateurs pseudodifférentiels*

Jean Nourrigat

*Laboratoire de mathématiques, EA4535 and FR.CNRS.3399, université de Reims, U.F.R. sciences exactes et naturelles, moulin de la Housse, BP 1039, 51687 Reims cedex 2, France*

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ABSTRACT

We determine the closure of the set of pseudodifferential operators of Calderón Vaillancourt type in the space of bounded linear operators in $L^2(\mathbb{R}^n)$, and also the closure of the similar classes of C. Rondeaux in the Schatten class. We give representation-theoretic characterizations of these classes.



RÉSUMÉ

On détermine l'adhérence de l'ensemble des opérateurs pseudodifférentiels appartenant à la classe de Calderón-Vaillancourt dans l'espace des opérateurs bornés dans $L^2(\mathbb{R}^n)$, et aussi des classes analogues de C. Rondeaux dans les classes de Schatten correspondantes. On donne une caractérisation de ces classes en termes de représentations de groupes.



## 1. Statement of the results

For each $p$ in $[1,\infty]$, let us denote by $W^{\infty p}(\mathbb{R}^{2n})$ the set of functions $F$ in $C^\infty(\mathbb{R}^{2n})$ which are in $L^p(\mathbb{R}^{2n})$ such as all their derivatives. For each function $F$ in $W^{\infty\infty}(\mathbb{R}^{2n})$, we denote by $Op(F)$ the operator formally defined, for each $f \in \mathcal{S}(\mathbb{R}^n)$, by:

$$\big(Op(F)f\big)(u) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi} F\left(\frac{x+y}{2}, \xi\right) f(y)\,\mathrm{d}y\,\mathrm{d}\xi \tag{1}$$

Calderón and Vaillancourt have shown in [3] that such an operator is well defined, and is bounded in $\mathcal{H} = L^2(\mathbb{R}^n)$. We denote by $\Psi_\infty(\mathcal{H})$ the space of operators $A$ in $\mathcal{L}(\mathcal{H})$ which are associated in this way to a symbol $F$ in $W^{\infty\infty}(\mathbb{R}^{2n})$.

We want to find the closure of $\Psi_\infty(\mathcal{H})$ in $\mathcal{L}(\mathcal{H})$, and to give a representation-theoretic formulation of the Beals characterization [2] of $\Psi_\infty(\mathcal{H})$.

We can ask the same question, replacing the set of bounded operators in $\mathcal{H}$ by one of the Schatten classes. For each $p$ in $[1,\infty[$, we denote by $\mathcal{L}^p(\mathcal{H})$ the Schatten class of operators $A$ in $\mathcal{L}(\mathcal{H})$ such that $|A|^p$ is trace class, where $|A| = (A^\star A)^{1/2}$. This space (see, for instance, [5]), is endowed with the norm:

$$\|A\|_p = \big(\mathrm{Tr}\big(|A|^p\big)\big)^{1/p} \tag{2}$$

*E-mail address:* jean.nourrigat@univ-reims.fr.

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It is proved in C. Rondeaux [4] that, for each $F$ in $W^{\infty p}(\mathbb{R}^{2n})$, the formal equality (1) still defines a bounded operator $Op(F)$ in $\mathcal{H} = L^2(\mathbb{R}^n)$, and that this operator is in the Schatten class $\mathcal{L}^p(\mathcal{H})$. Let us denote by $\Psi_p(\mathcal{H})$ the set of operators in $\mathcal{L}^p(\mathcal{H})$ that are defined in this way.

We denote by $H_n$ the Heisenberg group with dimension $2n+1$, i.e. $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ endowed with the composition law defined, for all $g = (X, Y, t)$ and $g' = (X', Y', t')$ in $H_n$, by:

$$g \circ g' = \left(X + X', Y + Y', t + t' + \frac{1}{2}\big(X \cdot Y' - Y \cdot X'\big)\right) \tag{3}$$

We shall denote by $\pi$ the representation of $H_n$ in $\mathcal{H} = L^2(\mathbb{R}^n)$ defined, for each $g = (X, Y, t)$ in $H_n$, and for each $f$ in $\mathcal{H}$, by:

$$\big(\pi(g)f\big)(u) = f(X+u)e^{i(u\cdot Y + t + \frac{1}{2}X\cdot Y)} \tag{4}$$

Let us agree that $\mathcal{L}^\infty(\mathcal{H}) = \mathcal{L}(\mathcal{H})$. For each $p$ in $[1, +\infty]$, we define also a representation $\Pi_p$ of $H_n$ in the Banach space $\mathcal{L}^p(\mathcal{H})$ by setting:

$$\Pi_p(g)(A) = \pi(g)A\pi(g)^{-1}, \quad g \in H_n,\ A \in \mathcal{L}^p(\mathcal{H}) \tag{5}$$

This representation is norm-preserving. If $p < \infty$, the representation $\Pi_p$ is continuous. In fact, if $A$ is of finite rank, we see easily that the map $g \to \Pi_p(g)(A)$ is continuous from $H_n$ to $\mathcal{L}^p(\mathcal{H})$. Then, we remark that the set of operators with finite rank is dense in $\mathcal{L}^p(\mathcal{H})$ if $p < +\infty$ (see [5]).

**Theorem 1.1.** a) *For each $p$ in $[1, +\infty]$, the set $\Psi_p(\mathcal{H})$ is the set of $C^\infty$ vectors of the representation $\Pi_p$, i.e. the set of operators $A$ in $\mathcal{L}^p(\mathcal{H})$ such that the map*

$$g \to \Pi_p(g)(A) \tag{6}$$

*is $C^\infty$ from $H_n$ to $\mathcal{L}^p(\mathcal{H})$.*

b) *The closure of $\Psi_\infty(\mathcal{H})$ in $\mathcal{L}(\mathcal{H})$ is the set of continuous vectors of the representation $\Pi_\infty$, i.e. the set of operators $A$ in $\mathcal{L}(\mathcal{H})$ such that the map* (6) *is continuous from $H_n$ to $\mathcal{L}(\mathcal{H})$. If $p < +\infty$, the set $\Psi_p(\mathcal{H})$ is dense in the Schatten class $\mathcal{L}^p(\mathcal{H})$.*

For $p = 1$, the point b) has been proved in [1].

## 2. Proof of point a) of Theorem 1.1

Let $P_j$ be the operator of derivation with respect to the variable $u_j$, and $Q_j$ be the operator of multiplication by $u_j$ ($1 \leqslant j \leqslant n$). For each operator $A$ in $\mathcal{L}(\mathcal{H})$, and for each multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$, the iterated commutator:

$$(\operatorname{ad} P)^\alpha (\operatorname{ad} Q)^\beta A = (\operatorname{ad} P_1)^{\alpha_1} \cdots (\operatorname{ad} P_n)^{\alpha_n} (\operatorname{ad} Q_1)^{\beta_1} \cdots (\operatorname{ad} Q_n)^{\beta_n} A \tag{7}$$

is well defined as an operator from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$. Let us recall the classical following result:

**Theorem 2.1.** *An operator $A$ in $\mathcal{L}^p(\mathcal{H})$ ($1 \leqslant p \leqslant +\infty$) is in $\Psi_p(\mathcal{H})$ if, and only if, for each multi-indices $\alpha$ and $\beta$, the operator $(\operatorname{ad} P)^\alpha (\operatorname{ad} Q)^\beta A$ (a priori defined as an operator from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$) is in $\mathcal{L}^p(\mathcal{H})$.*

If $p = +\infty$, Theorem 2.1 is a very classical result, the Beals characterization of pseudo-differential operators [2]. If $p < +\infty$, it has been proved in C. Rondeaux [4].

Let $\Pi$ be the representation of $H_n$ in $\mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$ defined as in (5), i.e. for each $g \in H_n$, for each $A$ in $\mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$, for each $\varphi$ and $\psi$ in $\mathcal{S}(\mathbb{R}^n)$:

$$\big\langle \Pi(g)(A)\varphi, \psi\big\rangle = \big\langle A\pi(g)^{-1}\varphi, \pi(g)^{-1}\psi\big\rangle \tag{8}$$

By definition (3), we see that, if $A$ is in $\mathcal{L}(\mathcal{H})$, if $\varphi$ and $\psi$ are in $\mathcal{S}(\mathbb{R}^n)$, both sides of this equality are $C^\infty$ functions in $H_n$, and that, for each $g = (X, Y, t)$ in $H_n$, we have:

$$\frac{\partial}{\partial X_j}\big\langle \Pi(X, Y, t)(A)\varphi, \psi\big\rangle = \big\langle \Pi(X, Y, t)\big([P_j, A]\big)\varphi, \psi\big\rangle \tag{9}$$

$$\frac{\partial}{\partial Y_j}\big\langle \Pi(X, Y, t)(A)\varphi, \psi\big\rangle = i\big\langle \Pi(X, Y, t)\big([Q_j, A]\big)\varphi, \psi\big\rangle \tag{10}$$

$$\frac{\partial}{\partial t}\big\langle \Pi(X, Y, t)(A)\varphi, \psi\big\rangle = 0 \tag{11}$$

We may iterate, and we shall use Taylor expansions of the left side of (8).

If $A$ is a $C^\infty$ vector of $\Pi_p$, we may replace $\Pi$ by $\Pi_p$ in the left side of (9), and equality (9), taken at the origin, shows that $[P_j, A]$ is in $\mathcal{L}^p(\mathcal{H})$, and that the following equality:

$$\frac{\partial \Pi_p(X,Y,t)(A)}{\partial X_j} = \Pi_p(X,Y,t)\big([P_j,A]\big) \tag{12}$$

is valid at the origin. Since $\Pi_p$ is a norm preserving representation, it follows that the same equality is valid for each $g$ in $H_n$. Since $A$ is a $C^\infty$ vector of the representation $\Pi_p$, it follows that $[P_j, A]$ is also a $C^\infty$ vector of $\Pi_p$. It is the same for $[Q_j, A]$, and we have:

$$\frac{\partial \Pi_p(X,Y,t)(A)}{\partial Y_j} = i\Pi_p(X,Y,t)\big([Q_j,A]\big), \qquad \frac{\partial \Pi_p(X,Y,t)(A)}{\partial t} = 0 \tag{13}$$

Therefore, we can iterate the same argument, replacing $A$ by these commutators, and prove by induction that all the commutators (7) are $C^\infty$ vectors of the representation $\Pi_p$, and in particular that they are in $\mathcal{L}^p(\mathcal{H})$. By Theorem 2.1, it follows that $A$ is in $\Psi_p(\mathcal{H})$.

Before proving the converse, we have to make clear a question of continuity for $p = +\infty$. Let $A$ be in $\mathcal{L}(\mathcal{H})$, such as the commutators $[P_j, A]$ and $[Q_j, A]$. We write a Taylor expansion of the left-hand side of (8). For each $g = (X, Y, t)$ in $H_n$, for each $\varphi$ and $\psi$ in $\mathcal{S}(\mathbb{R}^n)$, we may write:

$$\big\langle \big(\Pi(X,Y,t)(A) - A\big)\varphi, \psi\big\rangle = \sum_{j=1}^{n} \big[X_j A_j(X,Y,t,A,\varphi,\psi) + Y_j A_j(X,Y,t,A,\varphi,\psi)\big]$$

where:

$$A_j(X,Y,t,A,\varphi,\psi) = \int_0^1 \big\langle \Pi(\theta X,\theta Y,\theta t)\big([P_j,A]\big)\varphi,\psi\big\rangle\,\mathrm{d}\theta$$

and where $B_j(g, A, \varphi, \psi)$ is defined in a similar way. We may replace $\Pi$ by $\Pi_\infty$ above. Since $\Pi_\infty$ is a norm preserving representation, it follows that $|A_j(g, A, \varphi, \psi)| \leqslant \|[P_j, A]\|_{\mathcal{L}(\mathcal{H})}\|\varphi\|_{\mathcal{H}}\|\psi\|_{\mathcal{H}}$ and similarly for $B_j(g, A, \varphi, \psi)$. Then it follows that the map $g \to \Pi_\infty(g)(A)$ is continuous from $H_n$ to $\mathcal{L}(\mathcal{H})$ at the origin, and, since $\Pi_p$ is norm preserving, this map is continuous in all $H_n$.

If $A$ is in $\Psi_p(\mathcal{H})$, Theorem 2.1 and the remark above show that $A$ and all the commutators (7) are continuous vectors of the representation $\Pi_p$. (The above remarks are needed only for $p = +\infty$.) Hence, the following functions:

$$A_{jk}(X,Y,t) = \int_0^1 \Pi_p(\theta X,\theta Y,\theta t)\big(\big[P_j,[P_k,A]\big]\big)\,\mathrm{d}\theta, \qquad B_{jk}(X,Y,t) = i\int_0^1 \Pi_p(\theta X,\theta Y,\theta t)\big(\big[P_j,[Q_k,A]\big]\big)\,\mathrm{d}\theta$$

$$C_{jk}(X,Y,t) = -\int_0^1 \Pi_p(\theta X,\theta Y,\theta t)\big(\big[Q_j,[Q_k,A]\big]\big)\,\mathrm{d}\theta$$

are continuous and bounded in $H_n$, with values in $\mathcal{L}^p(\mathcal{H})$. The function $L$ defined in $H_n$ by

$$L(X,Y,t) = \Pi_p(X,Y,t)(A) - A - \sum_{j=1}^{n}\big[X_j\Pi_p\big([P_j,A]\big) + iY_j\Pi_p\big([Q_j,A]\big)\big]$$
$$- \frac{1}{2}\sum_{1\leqslant j,k\leqslant n}\big[X_jX_kA_{jk}(X,Y,t) + 2X_jY_kB_{jk}(X,Y,t) + Y_jY_kC_{jk}(X,Y,t)\big]$$

is also continuous in $H_n$, with values in $\mathcal{L}^p(\mathcal{H})$. For each $\varphi$ and $\psi$ in $\mathcal{S}(\mathbb{R}^n)$, we may write a Taylor expansion of the left-hand side of (8), which is $C^\infty$, using (9), (10) and (11), and replacing $\Pi$ by $\Pi_p$. We get $\langle L(X, Y, t)\varphi, \psi\rangle = 0$ for all $\varphi$ and $\psi$ in $\mathcal{S}(\mathbb{R}^n)$, and therefore $L(X, Y, t) = 0$. It follows that the map (6), from $H_n$ to $\mathcal{L}^p(\mathcal{H})$, is differentiable at the origin, and that its partial derivatives are given by (12) and (13) at the origin. Since $\Pi_p$ is a norm preserving representation, the map $g \to \Pi_p(g)(A)$ is differentiable in $H_n$, and its partial derivatives are still given by (12) and (13). By the above remark, the map (6) is $C^1$. By iterating, we see that this map is $C^\infty$ from $H_n$ to $\mathcal{L}^p(\mathcal{H})$. Point a) of Theorem 1.1 is proved.

## 3. Proof of point b) of Theorem 1.1

Let $A$ be an operator in $\mathcal{L}^p(\mathcal{H})$ ($1 \leqslant p \leqslant +\infty$) which is the limit, in $\mathcal{L}^p(\mathcal{H})$, of a sequence $(A_j)$ of operators in $\Psi_p(\mathcal{H})$. By the point a) of Theorem 1.1, the functions $g \to \Pi_p(g)(A_j)$ are $C^\infty$ from $H_n$ to $\mathcal{L}^p(\mathcal{H})$. We have, for each $g$ in $H_n$:

$$\left\| \Pi_p(g)(A_j - A) \right\|_{\mathcal{L}^p(\mathcal{H})} \leqslant \| A_j - A \|_{\mathcal{L}^p(\mathcal{H})} \tag{14}$$

Therefore, the function (6), being a uniform limit, in $H_n$, of a sequence of $C^\infty$ functions with values in $\mathcal{L}^p(\mathcal{H})$, is itself continuous from $H_n$ to $\mathcal{L}^p(\mathcal{H})$.

Conversely, let $A$ be a continuous vector of the representation $\Pi_p$. In order to give a suitable approximation of $A$, we define, for each $\lambda > 0$, the following operator:

$$\mathcal{T}_\lambda^{(p)} A = (\pi\lambda)^{-n} \int_{\mathbb{R}^{2n}} e^{-\frac{|X|^2+|Y|^2}{\lambda}} \Pi_p(X, Y, 0)(A)\, dX\, dY$$

This operator was already used in [1] (Section 5). We remark that:

$$\Pi_p(X, Y, t)\big(\mathcal{T}_\lambda^{(p)} A\big) = (\pi\lambda)^{-n} \int_{\mathbb{R}^{2n}} e^{-\frac{|X'-X|^2+|Y'-Y|^2}{\lambda}} \Pi_p\big(X', Y', 0\big)(A)\, dX'\, dY'$$

Then, it is clear that the function $g \to \Pi_p(g)(\mathcal{T}_\lambda^{(p)} A)$ is $C^\infty$ from $H_n$ to $\mathcal{L}^p(\mathcal{H})$. By point a) of Theorem 1.1, $\mathcal{T}_\lambda^{(p)} A$ is an element of $\Psi_p(\mathcal{H})$. Since $A$ is a continuous vector of $\Pi_p$, we have:

$$\lim_{\lambda \to 0} \left\| \mathcal{T}_\lambda^{(p)} A - A \right\|_{\mathcal{L}^p(\mathcal{H})} = 0$$

The proof is similar to that of the analogous elementary result for convolutions. It follows that the closure of $\Psi_p(\mathcal{H})$ in $\mathcal{L}^p(\mathcal{H})$ is the set of continuous vectors of $\Pi_p$. If $p < +\infty$, this set is all $\mathcal{L}^p(\mathcal{H})$.

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