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Homotopy of EVII

Homotopie d'EVII

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ABSTRACT

We determine explicitly some homotopy groups of the exceptional hermitian symmetric space $\mathrm{EVII} = \mathrm{E}_7/(S^1\mathrm{E}_6)$.

RÉSUMÉ

Nous déterminons explicitement quelques groupes d'homotopie de l'espace symétrique hermitien de type exceptionnel $\mathrm{EVII} = \mathrm{E}_7/(S^1\mathrm{E}_6)$.

1. The result

The goal of this Note is to calculate explicitly some lower homotopy groups of the exceptional hermitian symmetric space $\mathrm{E}_7/(S^1\mathrm{E}_6)$, in É. Cartan's notation EVII. The first eight homotopy groups of $\mathrm{EVII} = \mathrm{E}_7/(S^1\mathrm{E}_6)$ have been determined by Burns (see [3, Prop. 4.8] and [4, Prop. 2.5]). We continue this list as follows:

Theorem. *The homotopy groups $\pi_j(\mathrm{E}_7/(S^1\mathrm{E}_6))$ with $9 \leqslant j \leqslant 17$ are:*

	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}	π_{16}	π_{17}
$\mathrm{E}_7/(S^1\mathrm{E}_6)$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	0

Remark. The homotopy groups π_9, π_{10} and π_{14} of $\mathrm{E}_7/(S^1\mathrm{E}_6)$ can also be directly read off from the long exact sequence of homotopy groups of coset spaces together with the corresponding homotopy groups of E_6 and E_7 that can be found in [8, p. 363].

2. The proof of the theorem

The main ingredient of our proof is the following equation:

$$\pi_{i+1}\big(\mathrm{E}_7/\big(S^1\mathrm{E}_6\big)\big) \cong \pi_i\big(\big(S^1\mathrm{E}_6\big)/\mathrm{F}_4\big) \quad \text{for } 0 \leqslant i \leqslant 16. \tag{1}$$

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In the proof of his famous periodicity theorem for the stable homotopy of classical matrix groups, Bott used Morse theory to obtain relations between homotopy groups of certain compact symmetric spaces (see [2]). Such relations were studied later by Burns, Mitchell, and Nagano (see [3,9,11,10,4]). In this context Eq. (1) can be found in [11, p. 74] with reference to [3]. Since $(S^1E_6)/F_4$ is an isotropy orbit of a minuscule coweight in the tangent space of $E_7/(S^1E_6)$ (see e.g. [1, p. 311]), Eq. (1) is also a special case of a result due to Mitchell (see [10, Thm. 7.1]). The necessary information on root systems of symmetric spaces with multiplicities and coefficients in the highest root can be found in [6, pp. 477f., 503, 532ff.].

Recall that for $i \geqslant 2$ the homotopy group $\pi_i(M)$ of a connected manifold M is the same as the homotopy group $\pi_i(\tilde{M})$ of its universal cover $\tilde{M}$. Moreover $\pi_i(M_1 \times M_2) \cong \pi_i(M_1) \times \pi_i(M_2)$ for connected manifolds M_1 and M_2. Since $\pi_i(\mathbb{R}) \cong 0$, Eq. (1) implies:

$$\pi_{i+1}\big(E_7/\big(S^1E_6\big)\big) \cong \pi_i(E_6/F_4), \quad 2 \leqslant i \leqslant 16. \tag{2}$$

Our theorem now follows from the relevant homotopy groups of E_6/F_4 which are determined in [5, p. 411] (see also [3, Prop. 4.5] and [4, Prop. 2.2], we refer to [13, p. 186] for homotopy groups of spheres):

	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}	π_{16}
E_6/F_4	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	0

Remark. Since $\mathbb{O}P^2$ is an isotropy orbit of a minuscule coweight in the tangent space of E_6/F_4 (see e.g. [1, p. 310f.]), Mitchell's result [10, Thm. 7.1] and Eq. (2) yield:

$$\pi_{i+2}\big(E_7/\big(S^1E_6\big)\big) \cong \pi_{i+1}(E_6/F_4) \cong \pi_i\big(\mathbb{O}P^2\big) \quad \text{for } 1 \leqslant i \leqslant 14$$

(see also [5]). This shows that some homotopy groups of $E_7/(S^1E_6)$ can be traced back to homotopy groups of the Cayley plane $\mathbb{O}P^2$, which are determined in [7, Thm. 7.2].

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