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Partial Differential Equations

A blowup result for the periodic NLS without gauge invariance

Un résultat d'explosion pour l'équation de Schrödinger non linéaire sans invariance de gauge dans le cas périodique

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ABSTRACT

In this Note, we prove a finite-time blowup result for the periodic nonlinear Schrödinger equation on $\mathbb{T}^d$ with nonlinearity $|u|^p$ for $p > 1$. In particular, our blowup result holds above the Strauss exponent. This is in contrast with the non-periodic setting, where global existence for small data is known above the Strauss exponent.

RÉSUMÉ

Dans cette Note, nous démontrons un résultat d'explosion en temps fini pour l'équation de Schrödinger non linéaire sur le tore $\mathbb{T}^d$ avec une non linéarité du type $|u|^p$, $p > 1$. En particulier, notre résultat d'explosion est vrai pour des puissances p plus grandes que l'exposant de Strauss. Cette situation est contraire au cas non périodique où l'on connaît que pour p supérieur à l'exposant de Strauss, le problème de Cauchy est globalement bien posé.

1. Introduction

In this Note, we prove a finite-time blowup result for the following periodic NLS:

$$\begin{cases} i\partial_t u + \Delta u = \lambda |u|^p, \\ u|_{t=0} = u_0 \in H^s(\mathbb{T}^d), \end{cases} \quad (x,t) \in \mathbb{T}^d \times \mathbb{R}, \ p > 1. \tag{1}$$

A similar result was proven by Ikeda and Wakasugi [6] on $\mathbb{R}^d$ below the short range exponent: $1 < p \leqslant 1 + \frac{2}{d}$. Recall [4] that, on $\mathbb{R}^d$, there is global existence for small data if $p > p_S$, where p_S is the Strauss exponent given by $p_S = (d + 2 + \sqrt{d^2 + 12d + 4})/(2d) > 1 + \frac{2}{d}$. However, dispersion is much weaker on $\mathbb{T}^d$ and such small data global existence is not known on $\mathbb{T}^d$.[1] Indeed, our blowup result on $\mathbb{T}^d$ holds even above the Strauss exponent (see Theorems 1.2 and 1.4). We conclude that, on $\mathbb{T}^d$, there is no small data global well-posedness for (1) even above the Strauss exponent. The main purpose of this Note is to present this sharp contrast of the behaviors of solutions on $\mathbb{T}^d$ and $\mathbb{R}^d$.

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[1] Recall that the proof of small data global existence on $\mathbb{R}^d$ above the Strauss exponent is based on the dispersion estimate, which does not hold on $\mathbb{T}^d$.

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First, let us define the notion of weak solutions as in [6]:

Definition 1.1. We say that u is a local *weak* solution to (1) on $[0, T)$ if $u \in L^p(\mathbb{T}^d \times [0, T))$ and

$$\int_0^T \int_{\mathbb{T}^d} u(-i\partial_t \phi + \Delta\phi)\,\mathrm{d}x\,\mathrm{d}t = i \int_{\mathbb{T}^d} u_0(x)\phi(x, 0)\,\mathrm{d}x + \lambda \int_0^T \int_{\mathbb{T}^d} |u|^p \phi\,\mathrm{d}x\,\mathrm{d}t, \tag{2}$$

for any $\phi \in C^\infty(\mathbb{T}^d \times [0, T))$ with $\phi(x, T) \equiv 0$. If $T > 0$ can be made arbitrarily large, then we say that u is a global weak solution.

Write $\mu := \int_{\mathbb{T}^d} u_0(x)\,\mathrm{d}x$ as $\mu = \mu_1 + i\mu_2$, where $\mu_1 = \mathrm{Re} \int_{\mathbb{T}^d} u_0(x)\,\mathrm{d}x$ and $\mu_2 = \mathrm{Im} \int_{\mathbb{T}^d} u_0(x)\,\mathrm{d}x$.

Theorem 1.2. *Let $1 < p < \infty$ and $\lambda \neq 0$. Suppose that $\lambda = \lambda_1 + i\lambda_2$ and $\mu = \mu_1 + i\mu_2$ satisfy*

$$\lambda_1\mu_2 < 0 \quad \text{or} \quad \lambda_2\mu_1 > 0. \tag{3}$$

If u is a global-in-time weak solution to NLS (1), then $u(x, t) = 0$ a.e. on $\mathbb{T}^d \times [0, \infty)$.

The proof of Theorem 1.2 follows that in [6]. On $\mathbb{T}^d$, the result holds for a much wider range of p due to the boundedness of the spatial domain. By uniqueness of the Cauchy problem, Theorem 1.2 states that there is a finite-time blowup. See Theorem 1.4. Note that only the sign condition (3) is imposed in Theorem 1.2. In particular, this blowup result holds for small data as well.

In the well-posedness theory of the Cauchy problem (1), we usually say that u is a solution to (1) if it satisfies the following Duhamel formulation:

$$u(t) = S(t)u_0 - i\lambda \int_0^t S(t - t')|u|^p(t')\,\mathrm{d}t', \tag{4}$$

where $S(t) = e^{it\Delta}$. For $s > \frac{d}{2}$ and $p \in 2\mathbb{N}$, one can prove local well-posedness of (4) in H^s by Sobolev embedding. In [1], Bourgain introduced the $X^{s,b}$-space given by the norm: $\|u\|_{X^{s,b}(\mathbb{T}^d\times\mathbb{R})} = \|\langle n\rangle^s \langle \tau + |n|^2\rangle^b \widehat{u}(n, \tau)\|_{\ell^2_n L^2_\tau(\mathbb{Z}^d\times\mathbb{R})}$ and proved local well-posedness of NLS with nonlinearity $|u|^{p-1}u$ in L^2 for $p = 3$ and $d = 1$, and in H^s, $s > 0$, for odd integers $p \leqslant 1 + \frac{4}{d}$ with $d = 1, 2$. His argument directly applies to (1) in certain cases. In particular, (1) is locally well-posed in L^2 for $p \leqslant 3$ and $d = 1$, and in H^s, $s > 0$, for even integers $p \leqslant 1 + \frac{4}{d}$ with $d = 1, 2$. There are also well-posedness results in higher dimensions or for other values of p. See [1–3,5]. One may prove an analogous statement to Proposition 1.3 below in these settings, but for simplicity of the presentation, we will not discuss them in this Note.

A natural question is; do these solutions to (4) satisfy the weak formulation (2)? The following proposition provides a positive answer:

Proposition 1.3. *Assume that one of the followings holds*: (i) $s > \frac{d}{2}$ *and* $p \in 2\mathbb{N}$, (ii) $s = 0$, $p \leqslant 3$, $d = 1$, *or* (iii) $s > 0$, $p \leqslant 1 + \frac{4}{d}$, $p \in 2\mathbb{N}$ *with* $d = 1, 2$. *Then, if $u \in C([0, T) : H^s)$ satisfies the Duhamel formulation* (4) *on $[0, T)$ for some $T > 0$, then it is a weak solution to* (1) *in the sense of Definition* 1.1.

Recall the following blowup alternative: if u is a solution in $C([0, T) : H^s)$, then either (a) there exists $\varepsilon > 0$ such that u can be extended to $[0, T + \varepsilon)$ or (b) $\lim_{t\nearrow T} \|u(t)\|_{H^s} = \infty$. In the second case, such T is called the maximal time of existence.

The well-posedness result under the condition (i), (ii), or (iii) in Proposition 1.3 sustains this blowup alternative, since the Cauchy problem (1) is subcritical under (i), (ii), and (iii). Hence, Theorem 1.2 and Proposition 1.3 yield the following conclusion:

Theorem 1.4. *Let $1 < p < \infty$. Assume the hypotheses in Theorem* 1.2 *and Proposition* 1.3. *Then, the maximal time T^* of existence is finite and we have* $\liminf_{t\nearrow T^*} \|u(t)\|_{H^s} = \infty$.

Remark 1.5. By integrating (1) and taking the real and imaginary parts, we obtain

$$\partial_t\mu_1(t) = \lambda_2 \int_{\mathbb{T}^d} |u(t)|^p\,\mathrm{d}x, \qquad \partial_t\mu_2(t) = -\lambda_1 \int_{\mathbb{T}^d} |u(t)|^p\,\mathrm{d}x,$$

where $\mu_1(t) = \mathrm{Re} \int_{\mathbb{T}^d} u(t)\,\mathrm{d}x$ and $\mu_2(t) = \mathrm{Im} \int_{\mathbb{T}^d} u(t)\,\mathrm{d}x$. Hence, by Theorem 1.2, we deduce that any global solution must satisfy the following space–time bound:

$$\int_0^\infty \int_{\mathbb{T}^d} |u(t)|^p \,\mathrm{d}x\,\mathrm{d}t \leqslant \min\left(\frac{\mu_2(0)}{\lambda_1}, -\frac{\mu_1(0)}{\lambda_2}\right) < \infty.$$

From this, we conclude that any global solution must go to 0 as $t \to \infty$ in some averaged sense.

2. Proofs of Theorem 1.2 and Proposition 1.3

Proof of Theorem 1.2. The proof is based on the test-function method by Zhang [7,8]. For simplicity of presentation, we only prove Theorem 1.2 when $\lambda_1\mu_2 < 0$. Without loss of generality, assume $\lambda_1 = \operatorname{Re}\lambda > 0$ and $\mu_2 = \operatorname{Im}\int_{\mathbb{T}} u_0(x)\,\mathrm{d}x < 0$. Let $\eta \in C^\infty([0,\infty))$ such that it is non-negative, supported on $[0,1)$, $\eta(t) \equiv 1$ on $[0, \frac{1}{2})$, and $\partial_t \eta \leqslant C$. With $\eta_R(t) = \eta(t/R^2)$, $R > 0$, let $\phi_R(x,t) = \eta_R(t)$, i.e. ϕ_R is independent of $x \in \mathbb{T}^d$. Define I_R by

$$\mathrm{I}_R := \operatorname{Re}\lambda \int_0^{R^2} \int_{\mathbb{T}^d} |u|^p \phi_R^{p'} \,\mathrm{d}x\,\mathrm{d}t,$$

where p' is the Hölder conjugate of p. Then, from (2) with the Hölder inequality, we have

$$\begin{aligned}
\mathrm{I}_R &= \operatorname{Im}\int_{\mathbb{T}^d} u_0(x)\,\mathrm{d}x + \operatorname{Re}\int_0^{R^2}\int_{\mathbb{T}^d} u\big(-i\partial_t\big(\phi_R^{p'}\big) + \Delta\big(\phi_R^{p'}\big)\big)\,\mathrm{d}x\,\mathrm{d}t \\
&< \frac{p'}{R^2} \int_{\frac{R^2}{2}}^{R^2} \int_{\mathbb{T}^d} \big|u(x,t)\big| \phi_R^{p'-1}(x,t) \left|\partial_t \eta\left(\frac{t}{R^2}\right)\right| \mathrm{d}x\,\mathrm{d}t \\
&\lesssim R^{-\frac{2}{p}} \left(\int_{\frac{R^2}{2}}^{R^2} \int_{\mathbb{T}^d} \big|u(x,t)\big|^p \phi_R^{p'}(x,t)\,\mathrm{d}x\,\mathrm{d}t \right)^{\frac{1}{p}} \lesssim R^{-\frac{2}{p}} \mathrm{I}_R^{\frac{1}{p}}.
\end{aligned}$$

It is at the second to the last inequality where the boundedness of the spatial domain played a crucial role. Thus, for $1 < p < \infty$, we have

$$\mathrm{I}_R \lesssim R^{-\frac{2}{p-1}} \leqslant C, \tag{5}$$

where C is independent of $R > 1$. Since $\phi_R(x,t) \equiv 1$ on $\mathbb{T}^d \times [0, \frac{R^2}{2})$, we have

$$\int_0^{\frac{R^2}{2}} \int_{\mathbb{T}^d} |u|^p \,\mathrm{d}x\,\mathrm{d}t \leqslant C < \infty, \quad \text{independent of } R > 1.$$

By Monotone Convergence Theorem, we conclude that $u \in L^p(\mathbb{T}^d \times [0,\infty))$. From Fatou's lemma with (5), we obtain that $\|u\|_{L^p(\mathbb{T}^d\times[0,\infty))} \lesssim \lim_{R\to\infty} \mathrm{I}_R^{\frac{1}{p}} = \lim_{R\to\infty} R^{-\frac{2}{p(p-1)}} = 0$. In particular, we conclude that $u(x,t) = 0$ a.e. on $\mathbb{T}^d \times [0,\infty)$. $\square$

Proof of Proposition 1.3. First, assume (i); $u_0 \in H^s(\mathbb{T})$, $s > \frac{d}{2}$ and $p \in 2\mathbb{N}$. Then, there exists a unique solution $u \in C([0,T); H^s)$, satisfying (4). By Sobolev embedding, we have $u \in L^p(\mathbb{T}^d \times [0,T))$. Write $u(t) = S(t)u_0 + \mathcal{N}(u)(t)$, where $\mathcal{N}(u)$ denotes the second term in the Duhamel formulation (4). First, we show that the linear part $S(t)u_0$ satisfies

$$\int_0^T \int_{\mathbb{T}^d} S(t)u_0(-i\partial_t\phi + \Delta\phi)\,\mathrm{d}x\,\mathrm{d}t = i\int_{\mathbb{T}^d} u_0(x)\phi(x,0)\,\mathrm{d}x. \tag{6}$$

Let $u_{0,n}$ be smooth functions converging to u_0 in H^s. Then, $S(t)u_{0,n}$, $n \in \mathbb{N}$, solves the linear Schrödinger equation: $i\partial_t u + \Delta u = 0$ and is smooth on $\mathbb{T}^d \times [0,T)$. Integrating by parts, we have

$$\int_0^T \int_{\mathbb{T}^d} S(t)u_{0,n}(-i\partial_t\phi + \Delta\phi)\,\mathrm{d}x\,\mathrm{d}t = i\int_{\mathbb{T}^d} u_{0,n}(x)\phi(x,0)\,\mathrm{d}x. \tag{7}$$

By the Hölder inequality and the unitarity of $S(t)$ on L^2, we have

$$\left|\int_0^T\int_{\mathbb{T}^d}\big(S(t)u_0 - S(t)u_{0,n}\big)(-i\partial_t\phi+\Delta\phi)\,\mathrm{d}x\,\mathrm{d}t\right| \leqslant \|u_0-u_{0,n}\|_{L^2}\big(\|\phi\|_{W_t^{1,1}L_x^2}+\|\phi\|_{L_t^1H_x^2}\big)\longrightarrow 0. \tag{8}$$

Similarly, the right-hand side of (7) converges to the right-hand side of (6). Hence, (6) holds.

Next, we consider the nonlinear part $\mathcal{N}(u)$. Let u_n be smooth functions on $\mathbb{T}^d\times[0,T)$ converging to u in $C_TH^s := C([0,T);H^s)$. Then, by the algebra property of H^s, $s>\frac{d}{2}$, and the unitarity of $S(t)$, we have

$$\big\|\mathcal{N}(u)-\mathcal{N}(u_n)\big\|_{C_TH^s}\lesssim T\big(\|u\|_{C_TH^s}^{p-1}+\|u_n\|_{C_TH^s}^{p-1}\big)\|u-u_n\|_{C_TH^s}\longrightarrow 0. \tag{9}$$

Let $v_n=\mathcal{N}(u_n)$. Then, v_n solves the inhomogeneous linear Schrödinger equation:

$$i\partial_t v_n+\Delta v_n=\lambda|u_n|^p.$$

Note that $v_n(x,0)\equiv 0$. Then, proceeding as in (8) with (9) and integrating by parts, we have

$$\begin{aligned}\int_0^T\int_{\mathbb{T}^d}\mathcal{N}(u)(-i\partial_t\phi+\Delta\phi)\,\mathrm{d}x\,\mathrm{d}t &= \lim_{n\to\infty}\int_0^T\int_{\mathbb{T}^d}v_n(-i\partial_t\phi+\Delta\phi)\,\mathrm{d}x\,\mathrm{d}t\\ &=\lim_{n\to\infty}\int_0^T\int_{\mathbb{T}^d}(i\partial_t v_n+\Delta v_n)\phi\,\mathrm{d}x\,\mathrm{d}t\\ &=\lim_{n\to\infty}\lambda\int_0^T\int_{\mathbb{T}^d}|u_n|^p\phi\,\mathrm{d}x\,\mathrm{d}t=\lambda\int_0^T\int_{\mathbb{T}^d}|u|^p\phi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}\tag{10}$$

The identity (2) follows from (6) and (10).

Next, we briefly discuss the cases (ii) and (iii). The argument for the linear part remains the same. Hence, it suffices to establish the convergence of the nonlinear part $\mathcal{N}(u_n)$ to $\mathcal{N}(u)$ as in (9). This follows from the following multilinear estimate due to Bourgain [1,3]:

$$\big\|\mathcal{N}(u)\big\|_{C_TH^s}\lesssim\big\|\mathcal{N}(u)\big\|_{X_T^{s,b}}\lesssim T^\theta\|u\|_{X_T^{s,b}}^p,\quad\text{for some } b>\frac{1}{2}$$

under the condition (ii) or (iii) in Proposition 1.3, where $X_T^{s,b}$ is a local-in-time version of $X^{s,b}$ on $[0,T)$. Then, we take smooth functions u_n on $\mathbb{T}^d\times[0,T)$ converging to u in $X^{s,b}\subset C_TH^s$. The rest follows as before. This completes the proof of Proposition 1.3. □

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