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Functions of perturbed tuples of self-adjoint operators

Fonctions d'uplets d'opérateurs autoadjoints perturbés

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A R T I C L E I N F O

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A B S T R A C T

We generalize earlier results of Aleksandrov and Peller (2010) [2,3], Aleksandrov et al. (2011) [6], Peller (1985) [13], Peller (1990) [14] to the case of functions of n-tuples of commuting self-adjoint operators. In particular, we prove that if a function f belongs to the Besov space $B^1_{\infty,1}(\mathbb{R}^n)$, then f is operator Lipschitz and we show that if f satisfies a Hölder condition of order α, then $\|f(A_1,\dots,A_n)-f(B_1,\dots,B_n)\|\leqslant \mathrm{const}\max_{1\leqslant j\leqslant n}\|A_j-B_j\|^\alpha$ for all n-tuples of commuting self-adjoint operators $(A_1,\dots,A_n)$ and $(B_1,\dots,B_n)$. We also consider the case of arbitrary moduli of continuity and the case when the operators A_j-B_j belong to the Schatten–von Neumann class $\boldsymbol{S}_p$.

R É S U M É

Dans cette Note nous généralisons des résultats de Aleksandrov et Peller (2010) [2,3], Aleksandrov et al. (2011) [6], Peller (1985) [13], Peller (1990) [14] en cas de fonctions d'opérateurs auto-adjoints et d'opérateurs normaux. Nous considérons le problème similaire pour les fonctions de n-uplets d'opérateurs auto-adjoints qui commutent. En particulier, nous démontrons que si f est une fonction de la classe de Besov $B^1_{\infty,1}(\mathbb{R}^n)$, alors elle est lipschitzienne opératorielle. En outre, nous montrons que si f appartient à l'espace de Hölder d'ordre α, alors $\|f(A_1,\dots,A_n)-f(B_1,\dots,B_n)\|\leqslant \mathrm{const}\max_{1\leqslant j\leqslant n}\|A_j-B_j\|^\alpha$ por tous n-uplets $(A_1,\dots,A_n)$ et $(B_1,\dots,B_n)$ d'opérateurs auto-adjoints qui commutent. Nous considérons aussi le cas de module de continuité arbitraire et le cas où les opérateurs A_j-B_j appartiennent à l'espace de Schatten–von Neumann $\boldsymbol{S}_p$.

Version française abrégée

Il est bien connu (voir [10]) qu'il y a des fonctions f lipschitziennes sur la droite réelle $\mathbb{R}$ qui ne sont pas *lipschitziennes opératorielles*, c'est-à-dire la condition $|f(x)-f(y)|\leqslant \mathrm{const}|x-y|$, $x,y\in\mathbb{R}$, n'implique pas que pour tous les opérateurs auto-adjoints A et B l'inégalité

$$\big\|f(A)-f(B)\big\|\leqslant \mathrm{const}\,\|A-B\|,$$

soit vraie. Dans [13] et [14] des conditions nécessaires et des conditions suffisantes sont données pour qu'une fonction f soit lipschitzienne opératorielle. En particulier, il est démontré dans [13] que pour qu'une fonction f soit lipschitzienne

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opératorielle il est nécessaire que f appartienne localement à l'espace de Besov $B^1_{11}(\mathbb{R})$. Cela implique aussi qu'une fonction lipschitzienne n'est pas nécessairement lipschitzienne opératorielle. D'autre part, il est démontré dans [13] et [14] que si f appartient à l'espace de Besov $B^1_{\infty 1}(\mathbb{R})$, alors la fonction f est lipschitzienne opératorielle.

Il se trouve que la situation change dramatiquement si l'on considère les fonctions de la classe $\Lambda_\alpha(\mathbb{R})$ de Hölder d'ordre α, $0 < \alpha < 1$. Il est démontré dans [1] et [2] que si $f \in \Lambda_\alpha(\mathbb{R})$, $0 < \alpha < 1$ (c'est-à-dire $|f(x) - f(y)| \leqslant \text{const}\,|x - y|^\alpha$), alors f doit être *hölderienne opératorielle d'ordre* α, c'est-à-dire

$$\big\| f(A) - f(B) \big\| \leqslant \text{const}\,\|A - B\|^\alpha.$$

Ce résultat était généralisé dans [2] pour les modules de continuité arbitraires.

Les résultats ci-dessus ont été géneralisés dans [5] et [6] au cas de fonctions d'opérateurs normaux.

Dans cette note nous considérons le cas de fonctions de n-uplets d'opérateurs auto-adjoints qui commutent. Il se trouve que les méthodes du travail [6] ne marchent pas dans cette situation. Supposons que f est une fonction bornée sur $\mathbb{R}^3$ dont la transformée de Fourier a un support compact. On peut montrer que comme dans le cas d'opérateurs normaux, les fonction $\mathfrak{D}_1 f$ et $\mathfrak{D}_3 f$ sur $\mathbb{R}^3 \times \mathbb{R}^3$ définies par

$$(\mathfrak{D}_1 f)(x, y) = \frac{f(x_1, x_2, x_3) - f(y_1, x_2, x_3)}{x_1 - y_1}, \qquad (\mathfrak{D}_3 f)(x, y) = \frac{f(y_1, y_2, x_3) - f(y_1, y_2, y_3)}{x_3 - y_3}$$

sont des multiplicateurs de Schur (voir §3 pour la définition). Toutefois, contrairement au cas $n = 2$, la fonction $\mathfrak{D}_2$ définie par

$$(\mathfrak{D}_2 f)(x, y) = \frac{f(y_1, x_2, x_3) - f(y_1, y_2, x_3)}{x_2 - y_2},$$

n'est pas un multiplicateur de Schur.

Cependant, nous démontrons le résultat suivant :

Soient $\sigma > 0$ et f une fonction dans $L^\infty(\mathbb{R}^n)$ dont la transformée de Fourier a un support dans $\{\xi \in \mathbb{R}^n\colon \|\xi\| \leqslant \sigma\}$. Alors il y a des fonctions Ψ_j, $1 \leqslant j \leqslant n$, sur $\mathbb{R}^n \times \mathbb{R}^n$ qui appartiennent à l'espace $\mathfrak{M}_{\mathbb{R}^n,\mathbb{R}^n}$ de multiplicateurs de Schur et telles que

$$f(x_1, \ldots, x_n) - f(y_1, \ldots, y_n) = \sum_{j=1}^n (x_j - y_j)\Psi_j(x_1, \ldots, x_n, y_1, \ldots, y_n), \quad x_j, y_j \in \mathbb{R},$$

et

$$\|\Psi_j\|_{\mathfrak{M}_{\mathbb{R}^n,\mathbb{R}^n}} \leqslant \text{const}\,\sigma\,\|f\|_{L^\infty(\mathbb{R}^n)}.$$

Ce résultat implique que si f appartient à l'espace de Besov $B^1_{\infty,1}(\mathbb{R}^n)$ (voir [12]), alors f est lipschitzienne opératorielle, c'est-à-dire

$$\big\| f(A_1, \ldots, A_n) - f(B_1, \ldots, B_n) \big\| \leqslant \text{const} \max_{1 \leqslant j \leqslant n} \|A_j - B_j\|$$

pour tous n-uplets $(A_1, \ldots, A_n)$ et $(B_1, \ldots, B_n)$ d'opérateurs auto-adjoints qui commutent.

Nous démontrons aussi que si f est une fonction hölderienne d'ordre α sur $\mathbb{R}^n$, alors f est une fonction hölderienne opératorielle d'ordre α, c'est-à-dire

$$\big\| f(A_1, \ldots, A_n) - f(B_1, \ldots, B_n) \big\| \leqslant \text{const} \max_{1 \leqslant j \leqslant n} \|A_j - B_j\|^\alpha$$

pour tous n-uplets $(A_1, \ldots, A_n)$ et $(B_1, \ldots, B_n)$ d'opérateurs autoadjoints qui commutent.

Nous obtenons aussi des analogues d'autres résultats de [13,14,2] et [3] pour les fonctions d'n-uplets d'opérateurs auto-adjoint qui commutent (voir la version anglaise).

1. Introduction

In this note we study the behavior of functions of perturbed tuples of commuting self-adjoint operators. We are going to find sharp estimates for $f(A_1, \ldots, A_n) - f(B_1, \ldots, B_n)$, where $(A_1, \ldots, A_n)$ and $(B_1, \ldots, B_n)$ are n-tuples of commuting self-adjoint operators and f is a function on $\mathbb{R}^n$. Our results generalize the results of [13,14,1–6] for self-adjoint and normal operators.

Recall that a Lipschitz function f on the real line $\mathbb{R}$ does not have satisfy the inequality

$$\big\| f(A) - f(B) \big\| \leqslant \text{const}\,\|A - B\|$$

for arbitrary self-adjoint operators A and B on Hilbert space, i.e., it does not have to be *operator Lipschitz*. This was proved in [10]. Later it was shown in [13] and [14] that if f is operator Lipschitz, then f locally belongs to the Besov space $B^1_{1,1}(\mathbb{R})$ (see [12] for an introduction to Besov spaces) which also implies that Lipschitzness is not sufficient for operator Lipschitzness. On the other hand, it was proved in [13] and [14] that if f belongs to the Besov space $B^1_{\infty,1}(\mathbb{R})$, then f is operator Lipschitz.

The situation changes dramatically if instead of the Lipschitz class, we consider the Hölder classes $\Lambda_\alpha(\mathbb{R})$, $0<\alpha<1$, of functions f satisfying the inequality $|f(x)-f(y)|\leqslant \text{const}\,|x-y|^\alpha$, $x,y\in\mathbb{R}$. It was shown in [1] and [2] that a function f in $\Lambda_\alpha(\mathbb{R})$ must be *operator Hölder of order* α, i.e.,

$$\big\|f(A)-f(B)\big\|\leqslant \text{const}\,\|A-B\|^\alpha,$$

for arbitrary self-adjoint operators A and B. Note that the papers [1] and [2] also contain sharp estimates of $\|f(A)-f(B)\|$ for functions f of class Λ_ω for arbitrary moduli of continuity ω.

It was also proved in [1] and [3] that if $f\in\Lambda_\alpha$, $p>1$, and A and B are self-adjoint operators such that $A-B$ belongs to the Schatten–von Neumann class $\boldsymbol{S}_p$, then $f(A)-f(B)\in\boldsymbol{S}_{p/\alpha}$ and

$$\big\|f(A)-f(B)\big\|_{\boldsymbol{S}_{p/\alpha}}\leqslant \text{const}\,\|A-B\|^\alpha_{\boldsymbol{S}_p}.$$

Later in [5] and [6] the above results were generalized to the case of functions of normal operators. Note that the proofs given in [13,14,1–3] for self-adjoint operators do not work in the case of normal operators and a new approach was used in [5] and [6].

In this paper we consider a more general problem of functions of n-tuples of commuting self-adjoint operators. The case $n=2$ corresponds to the case of normal operators. It turns out that the techniques used in [6] do not work for $n\geqslant 3$. We offer in this note a new approach that works for all $n\geqslant 1$.

We are going to use the technique of double operator integrals developed in [7–9]. Double operator integrals are expressions of the form

$$\iint\limits_{\mathcal{X}_1\times\mathcal{X}_2}\Phi(s_1,s_2)\,\mathrm{d}E_1(s_1)T\,\mathrm{d}E_2(s_2),\tag{1}$$

where E_1 and E_2 are spectral measures on $\mathcal{X}_1$ and $\mathcal{X}_2$, Φ is a bounded measurable function on $\mathcal{X}_1\times\mathcal{X}_2$, and T is an operator on Hilbert space. It was observed in [7–9] that the double operator integral (1) is well defined if $T\in\boldsymbol{S}_2$ and determines an operator of class $\boldsymbol{S}_2$. For certain Φ, the transformer $T\mapsto\iint\Phi\,\mathrm{d}E_1T\,\mathrm{d}E_2$ maps the trace class $\boldsymbol{S}_1$ into itself. If so, one can define by duality the integral (1) for all bounded operators T. Such functions Φ are called *Schur multipliers* (with respect to the spectral measures E_1 and E_2). We refer the reader to [13] for characterizations of Schur multipliers.

If $\mathcal{X}_1$ and $\mathcal{X}_2$ are Borel subsets of Euclidean spaces, *we use the notation* $\mathfrak{M}_{\mathcal{X}_1,\mathcal{X}_2}$ *for the space of Borel functions* Φ *on* $\mathcal{X}_1\times\mathcal{X}_2$ *that are Schur multipliers for all Borel spectral measures* E_1 *and* E_2 *on* $\mathcal{X}_1$ *and* $\mathcal{X}_2$.

The proofs of the results of [6] for normal operators were based on the following formula:

$$\begin{aligned}f(N_1)-f(N_2)&=\iint(\mathfrak{D}_yf)(z_1,z_2)\,\mathrm{d}E_1(z_1)(B_1-B_2)\,\mathrm{d}E_2(z_2)\\&\quad+\iint(\mathfrak{D}_xf)(z_1,z_2)\,\mathrm{d}E_1(z_1)(A_1-A_2)\,\mathrm{d}E_2(z_2).\end{aligned}$$

Here N_1 and N_2 are normal operators with bounded difference N_1-N_2, $A_j=\operatorname{Re}N_j$, $B_j=\operatorname{Im}N_j$, $x_j=\operatorname{Re}z_j$, $y_j=\operatorname{Im}z_j$, f is a bounded function on $\mathbb{R}^2$ whose Fourier transform has compact support,

$$(\mathfrak{D}_xf)(z_1,z_2)=\frac{f(x_1,y_2)-f(x_2,y_2)}{x_1-x_2},\quad\text{and}\quad(\mathfrak{D}_yf)(z_1,z_2)=\frac{f(x_1,y_1)-f(x_1,y_2)}{y_1-y_2},\quad z_1,z_2\in\mathbb{C}.$$

It was shown in [6] that $\mathfrak{D}_xf$ and $\mathfrak{D}_yf$ belong to the space of Schur multipliers $\mathfrak{M}_{\mathbb{R}^2,\mathbb{R}^2}$.

However, in the case $n\geqslant 3$ the situation is more complicated. Let (A_1,A_2,A_3) and (B_1,B_2,B_3) be triples of commuting self-adjoint operators. Suppose that f is a bounded function on $\mathbb{R}^3$ whose Fourier transform has compact support. It can be shown that

$$\begin{aligned}f(A_1,A_2,A_3)-f(B_1,B_2,B_3)&=\iint(\mathfrak{D}_1f)(x,y)\,\mathrm{d}E_1(x)(A_1-B_1)\,\mathrm{d}E_2(y)\\&\quad+\iint(\mathfrak{D}_2f)(x,y)\,\mathrm{d}E_1(x)(A_2-B_2)\,\mathrm{d}E_2(y)\\&\quad+\iint(\mathfrak{D}_3f)(x,y)\,\mathrm{d}E_1(x)(A_3-B_3)\,\mathrm{d}E_2(y),\end{aligned}$$

whenever the functions $\mathfrak{D}_1f$, $\mathfrak{D}_2f$, and $\mathfrak{D}_3f$ belong to the space of Schur multipliers $\mathfrak{M}_{\mathbb{R}^3,\mathbb{R}^3}$. Here

$$(\mathfrak{D}_1 f)(x,y) = \frac{f(x_1,x_2,x_3) - f(y_1,x_2,x_3)}{x_1 - y_1}, \qquad (\mathfrak{D}_2 f)(x,y) = \frac{f(y_1,x_2,x_3) - f(y_1,y_2,x_3)}{x_2 - y_2},$$
$$(\mathfrak{D}_3 f)(x,y) = \frac{f(y_1,y_2,x_3) - f(y_1,y_2,y_3)}{x_3 - y_3}, \quad x = (x_1,x_2,x_3),\ y = (y_1,y_2,y_3).$$

The methods of [6] allow us to prove that $\mathfrak{D}_1 f$ and $\mathfrak{D}_3 f$ do belong to the space of Schur multipliers $\mathfrak{M}_{\mathbb{R}^3,\mathbb{R}^3}$. However, as the next result shows, the function $\mathfrak{D}_2 f$ does not have to be in $\mathfrak{M}_{\mathbb{R}^3,\mathbb{R}^3}$.

Theorem 1.1. *Suppose that g is a bounded function on $\mathbb{R}$ such that the Fourier transform of g has compact support and is not a measure. Let f be the function on $\mathbb{R}^3$ defined by*

$$f(x_1,x_2,x_3) = g(x_1 - x_3)\sin x_2.$$

Then $\mathfrak{D}_2 f \notin \mathfrak{M}_{\mathbb{R}^3,\mathbb{R}^3}$.

Note that it is easy to construct such a function g, e.g., $g(x) = \int_0^x t^{-1}\sin t\,\mathrm{d}t$.

In Section 2 we show that in the case $n \geqslant 3$ it is possible to represent $f(A_1,\dots,A_n) - f(B_1,\dots,B_n)$ in terms of double operator integrals in a different way. Using such a representation, we obtain in Section 3 and Section 4 analogs of the above results in the case of n-tuples of commuting self-adjoint operators.

2. An integral representation

The integral representation for $f(A_1,\dots,A_n) - f(B_1,\dots,B_n)$ is based on the following result:

Theorem 2.1. *Let $\sigma > 0$ and let f be a function in $L^\infty(\mathbb{R}^n)$ whose Fourier transform is supported on $\{\xi \in \mathbb{R}^n\colon\ \|\xi\| \leqslant \sigma\}$. Then there exist functions Ψ_j in $\mathfrak{M}_{\mathbb{R}^n,\mathbb{R}^n}$, $1 \leqslant j \leqslant n$, such that*

$$f(x_1,\dots,x_n) - f(y_1,\dots,y_n) = \sum_{j=1}^{n} (x_j - y_j)\Psi_j(x_1,\dots,x_n,y_1,\dots,y_n), \quad x_j, y_j \in \mathbb{R}, \tag{2}$$

and $\|\Psi_j\|_{\mathfrak{M}_{\mathbb{R}^n,\mathbb{R}^n}} \leqslant \operatorname{const}\sigma\|f\|_{L^\infty(\mathbb{R}^n)}$.

We are going to derive Schur multiplier estimates from the following lemma.

Lemma 2.2. *Let $\mathcal{C} = \mathcal{Q} \times \mathcal{R}$ be a cube in $\mathbb{R}^{2n}$ of sidelength L and let Ψ be a C^∞ function on $\frac{3}{2}\mathcal{C}$. Then $\Psi|\mathcal{C} \in \mathfrak{M}_{\mathcal{Q},\mathcal{R}}$ and*

$$\|\Psi\|_{\mathfrak{M}_{\mathcal{Q},\mathcal{R}}} \leqslant \operatorname{const}\max\Big\{L^{|\alpha|}\max_{a\in\frac{3}{2}\mathcal{C}}\big|\big(D^\alpha\Psi\big)(a)\big|\colon\ |\alpha| \leqslant 2n+2\Big\}.$$

The lemma can be proved by expanding Ψ in the Fourier series.

Sketch of the proof of Theorem 2.1. By rescaling, we may assume that $\|f\|_{L^\infty} \leqslant 1$ and $\sigma = 1$.

We consider the lattice of dyadic cubes in $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$, i.e., the cubes whose sides are intervals of the form $[j2^k, (j+1)2^k)$, $j,k \in \mathbb{Z}$. We say that a dyadic cube $\mathcal{C}$ in $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ is *admissible* if either its sidelength $\mathrm{L}(\mathcal{C})$ is equal to 1 or $\mathrm{L}(\mathcal{C}) > 1$ and the interior of the cube $2\mathcal{C}$, i.e., the cube centered at the center of $\mathcal{C}$ with sidelength $2\mathrm{L}(\mathcal{C})$, does not intersect the diagonal $\{(x,x)\colon\ x \in \mathbb{R}^n\}$. An admissible cube is called *maximal* if it is not a proper subset of another admissible cube. It is easy to see that the maximal admissible cubes are disjoint and cover $\mathbb{R}^{2n}$. It can also easily be verified that if $\mathcal{Q}$ is a dyadic cube in $\mathbb{R}^n$, then there can be at most 6^n dyadic cubes $\mathcal{R}$ in $\mathbb{R}^n$ such that $\mathcal{Q} \times \mathcal{R}$ is a maximal admissible cube. For $l = 2^m$, we denote by $\mathcal{D}_l$ the set of maximal dyadic cube of sidelength l.

It follows that if Ω is a function on $\mathbb{R}^n \times \mathbb{R}^n$ that is supported on $\bigcup_{\mathcal{C}\in\mathcal{D}_l}\mathcal{C}$, then

$$\|\Omega\|_{\mathfrak{M}_{\mathbb{R}^n,\mathbb{R}^n}} \leqslant 6^n \sup_{\mathcal{C}\in\mathcal{D}_l}\|\chi_{\mathcal{C}}\Omega\|_{\mathfrak{M}_{\mathbb{R}^n,\mathbb{R}^n}}.$$

We have to define Ψ_j on each maximal admissible cube. Suppose that $\mathcal{C} \in \mathcal{D}_1$. We put

$$\Psi_j(x,y) = \int_0^1 (D_j f)\big((1-t)x + ty\big)\,\mathrm{d}t, \quad (x,y) \in \mathcal{C} = \mathcal{Q} \times \mathcal{R},$$

where $D_j f$ is the jth partial derivative of f. It follows from Lemma 2.2 that $\|\Psi_j\|_{\mathfrak{M}_{\mathcal{Q},\mathcal{R}}} \leqslant \operatorname{const}$.

Suppose now that $l = 2^m > 1$ and $\mathcal{C} = \mathcal{Q} \times \mathcal{R} \in \mathcal{D}_l$. Let ω be a C^∞ nonnegative even function on $\mathbb{R}$ such that $\omega(t) = 0$ for $t \in [-\frac{1}{2}, \frac{1}{2}]$, and $\omega(t) = 1$ for $t \notin [-1, 1]$. We put $\Phi_j(x, y) = \omega((x_j - y_j)/l)$, $\Phi = \sum_{j=1}^n \Phi_j$, and define the functions Ξ_j, $1 \leqslant j \leqslant n$, by

$$\Xi_j(x, y) = \begin{cases} \frac{1}{x_j - y_j} \cdot \frac{\Phi_j(x,y)}{\Phi(x,y)}, & x_j \neq y_j, \\ 0, & x_j = y_j. \end{cases}$$

It follows easily from Lemma 2.2 that $\|\Xi_j\|_{\mathfrak{M}_{\mathcal{Q},\mathcal{R}}} \leqslant \operatorname{const} 2^{-m}$. We put now

$$\Psi_j(x, y) = \big(f(x) - f(y)\big)\Xi_j(x, y), \quad (x, y) \in \mathcal{C}.$$

Clearly, (2) holds for $(x, y) \in \mathcal{C}$ and $\|\Psi_j\|_{\mathfrak{M}_{\mathcal{Q},\mathcal{R}}} \leqslant \operatorname{const} 2^{-m}$. The functions Ψ_j are now defined on $\mathbb{R}^n \times \mathbb{R}^n$ and $\|\Psi_j\|_{\mathfrak{M}_{\mathbb{R}^n,\mathbb{R}^n}} \leqslant \operatorname{const} \sum_{m \geqslant 0} 2^{-m}$. This implies the result. □

Theorem 2.3. *Let f be a function satisfying the hypotheses of Theorem 2.1 and let Ψ_j, $1 \leqslant j \leqslant n$, be functions in $\mathfrak{M}_{\mathbb{R}^n,\mathbb{R}^n}$ satisfying (2). Suppose that $(A_1, \dots, A_n)$ and $(B_1, \dots, B_n)$ are n-tuples of commuting self-adjoint operators such that the operators $A_j - B_j$ are bounded, $1 \leqslant j \leqslant n$. Then*

$$f(A_1, \dots, A_n) - f(B_1, \dots, B_n) = \sum_{j=1}^n \iint\limits_{\mathbb{R}^n \times \mathbb{R}^n} \Psi_j(x, y)\, \mathrm{d}E_A(x)(A_j - B_j)\, \mathrm{d}E_B(y)$$

and $\|f(A_1, \dots, A_n) - f(B_1, \dots, B_n)\| \leqslant \operatorname{const} \sigma \|f\|_{L^\infty(\mathbb{R}^n)} \max_{1 \leqslant j \leqslant n} \|A_j - B_j\|$.

3. Operator norm estimates

In this section we obtain operator norm estimates for $f(A_1, \dots, A_n) - f(B_1, \dots, B_n)$, where $(A_1, \dots, A_n)$ and $(B_1, \dots, B_n)$ are n-tuples of commuting self-adjoint operators.

A function f on $\mathbb{R}^n$ is called *operator Lipschitz* if

$$\big\|f(A_1, \dots, A_n) - f(B_1, \dots, B_n)\big\| \leqslant \operatorname{const} \max_{1 \leqslant j \leqslant n} \|A_j - B_j\|$$

for all n-tuples of commuting self-adjoint operators $(A_1, \dots, A_n)$ and $(B_1, \dots, B_n)$.

The following theorem can be deduced easily from Theorem 2.3:

Theorem 3.1. *Let f be a function in the Besov space $B^1_{\infty,1}(\mathbb{R}^n)$. Then f is operator Lipschitz.*

For $\alpha \in (0, 1)$, we define the Hölder class $\Lambda_\alpha(\mathbb{R}^n)$ of functions f on $\mathbb{R}^n$ such that

$$\big|f(x) - f(y)\big| \leqslant \operatorname{const} \|x - y\|_{\mathbb{R}^n}^\alpha, \quad x, y \in \mathbb{R}^n.$$

For a modulus of continuity ω, the space $\Lambda_\omega(\mathbb{R}^n)$ consists of functions f on $\mathbb{R}^n$ such that

$$\big|f(x) - f(y)\big| \leqslant \operatorname{const} \omega\big(\|x - y\|_{\mathbb{R}^n}\big), \quad x, y \in \mathbb{R}^n.$$

The following results are analogs of the corresponding results of [1] and [2] in the case $n = 1$. The proofs of Theorems 3.2 and 3.3 are based on Theorem 2.3 and use the same methods as in [2].

Theorem 3.2. *Let $\alpha \in (0, 1)$ and let $f \in \Lambda_\alpha(\mathbb{R}^n)$. Then f is operator Hölder of order α, i.e.,*

$$\big\|f(A_1, \dots, A_n) - f(B_1, \dots, B_n)\big\| \leqslant \operatorname{const} \max_{1 \leqslant j \leqslant n} \|A_j - B_j\|^\alpha$$

for all n-tuples of commuting self-adjoint operators $(A_1, \dots, A_n)$ and $(B_1, \dots, B_n)$.

Theorem 3.3. *Let ω be a modulus of continuity and let $f \in \Lambda_\omega(\mathbb{R}^n)$. Then*

$$\big\|f(A_1, \dots, A_n) - f(B_1, \dots, B_n)\big\| \leqslant \operatorname{const} \omega_*\Big(\max_{1 \leqslant j \leqslant n} \|A_j - B_j\|\Big)$$

for all n-tuples of commuting self-adjoint operators $(A_1, \dots, A_n)$ and $(B_1, \dots, B_n)$, where

$$\omega_*(\delta) \stackrel{\text{def}}{=} \delta \int\limits_\delta^\infty \frac{\omega(t)}{t^2}\, \mathrm{d}t, \quad \delta > 0.$$

4. Schatten–von Neumann norm estimates

In this section we obtain estimates in $\boldsymbol{S}_p$ norms.

Theorem 4.1. *Let f be a function in the Besov space $B^1_{\infty,1}(\mathbb{R}^n)$. Suppose that $(A_1,\dots,A_n)$ and $(B_1,\dots,B_n)$ are n-tuples of commuting self-adjoint operators such that $A_j - B_j \in \boldsymbol{S}_1$. Then $f(A_1,\dots,A_n) - f(B_1,\dots,B_n) \in \boldsymbol{S}_1$ and*

$$\big\| f(A_1,\dots,A_n) - f(B_1,\dots,B_n)\big\|_{\boldsymbol{S}_1} \leqslant \mathrm{const}\,\|f\|_{B^1_{\infty,1}(\mathbb{R}^n)} \max_{1\leqslant j\leqslant n} \|A_j - B_j\|_{\boldsymbol{S}_1}.$$

Theorem 4.2. *Let $f \in \Lambda_\alpha(\mathbb{R}^n)$ and let $p > 1$. Suppose that $(A_1,\dots,A_n)$ and $(B_1,\dots,B_n)$ are n-tuples of commuting self-adjoint operators such that $A_j - B_j \in \boldsymbol{S}_p$. Then $f(A_1,\dots,A_n) - f(B_1,\dots,B_n) \in \boldsymbol{S}_{p/\alpha}$ and*

$$\big\| f(A_1,\dots,A_n) - f(B_1,\dots,B_n)\big\|_{\boldsymbol{S}_{p/\alpha}} \leqslant \mathrm{const}\,\|f\|_{\Lambda_\alpha(\mathbb{R}^n)} \max_{1\leqslant j\leqslant n} \|A_j - B_j\|^\alpha_{\boldsymbol{S}_p}.$$

Note that the conclusion of Theorem 4.2 does not hold in the case $p = 1$ even if $n = 1$, see [3].

Theorem 4.3. *Let f be a function in the Besov space $B^\alpha_{\infty,1}(\mathbb{R}^n)$. Suppose that $(A_1,\dots,A_n)$ and $(B_1,\dots,B_n)$ are n-tuples of commuting self-adjoint operators such that $A_j - B_j \in \boldsymbol{S}_1$. Then $f(A_1,\dots,A_n) - f(B_1,\dots,B_n) \in \boldsymbol{S}_{1/\alpha}$ and*

$$\big\| f(A_1,\dots,A_n) - f(B_1,\dots,B_n)\big\|_{\boldsymbol{S}_{1/\alpha}} \leqslant \mathrm{const}\,\|f\|_{B^\alpha_{\infty,1}(\mathbb{R}^n)} \max_{1\leqslant j\leqslant n} \|A_j - B_j\|^\alpha_{\boldsymbol{S}_1}.$$

The proofs of the above theorems are based on Theorem 2.3 and use the methods of [3].

Note that in [3] more general results for other operator ideals were obtained in the case $n = 1$. Those results can also be generalized to the case of arbitrary $n \geqslant 1$.

We would like to mention the paper [11] on Lipschitz estimates in the norm of $\boldsymbol{S}_p$, $1 < p < \infty$, for functions of commuting tuples of self-adjoint operators.

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