



Partial Differential Equations

## Optimal Hardy-type inequalities for elliptic operators

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## ABSTRACT

For a general second order elliptic operator  $P$  in a domain  $\Omega$ , we construct a Hardy weight  $W$  in the punctured domain  $\Omega^* := \Omega \setminus \{0\}$  such that  $P - \lambda W$  is subcritical in  $\Omega^*$  for  $\lambda < 1$ , null-critical in  $\Omega^*$  for  $\lambda = 1$ , and supercritical near infinity and near 0 for  $\lambda > 1$ . Our method is based on the theory of positive solutions and applies to both symmetric and nonsymmetric operators. The constructed Hardy weight is given by an explicit formula involving the Green function of  $P$  and its gradient.

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## R É S U M É

Soit  $P$  un opérateur elliptique du second ordre sur un domaine  $\Omega$ . On construit un poids  $W$ , tel que si  $\Omega^* := \Omega \setminus \{0\}$  est un domaine épointé, alors  $P - \lambda W$  est sous-critique sur  $\Omega^*$  pour  $\lambda < 1$ , nul-critique dans  $\Omega^*$  pour  $\lambda = 1$ , et supercritique à l'infini et en 0 pour  $\lambda > 1$ . Notre approche repose sur la théorie des solutions positives d'un opérateur elliptique du second ordre, et s'applique à la fois au cas symétrique et non symétrique. Le poids est de plus donné par une formule explicite faisant intervenir la fonction de Green de  $P$  et son gradient.

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## Version française abrégée

Le but principal de cette Note est d'obtenir des inégalités de Hardy, c'est-à-dire des généralisations de l'inégalité de Hardy classique dans  $\mathbb{R}^n \setminus \{0\}$ ,  $n \geq 2$  :

$$\int_{\mathbb{R}^n \setminus \{0\}} |\nabla u|^2 dx \geq C_H \int_{\mathbb{R}^n \setminus \{0\}} \frac{u^2(x)}{|x|^2} dx \quad \forall u \in C_0^\infty(\mathbb{R}^n \setminus \{0\}), \quad (1)$$

où  $C_H = (\frac{n-2}{2})^2$ . Outre l'inégalité en elle-même, les faits suivants sont importants :

- (a) Le potentiel  $\frac{C_H}{|x|^2}$  est *critique*. En particulier, la constante  $C_H$  est optimale ;
- (b) La constante  $C_H$  est optimale à l'infini et en 0 ;
- (c) L'infimum du problème variationnel associé n'est pas réalisé.

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De façon générale, étant donné un opérateur elliptique symétrique du second ordre  $P$ , associé à une forme quadratique  $q$  sur un domaine  $\Omega$ , on dit qu'une inégalité de Hardy est vérifiée lorsque

$$q(u, u) \geq \int_{\Omega} W u^2 \, dx \quad \forall u \in C_0^\infty(\Omega), \tag{2}$$

où  $W$  est un poids positif non-nul sur  $\Omega$ . L'existence d'un tel  $W$  est équivalente à la sous-criticité de  $P$ . Bien sûr, si  $P$  est sous-critique, un grand nombre de poids  $W$  satisfont (1). En fait, l'ensemble des tels poids  $W$  est alors un convexe non-compact dont les points extrémaux sont les poids dits *critiques*. Cependant, que le poids soit critique ne suffit pas du tout pour avoir une inégalité de Hardy ayant les mêmes propriétés que (1) : par exemple, des poids critiques peuvent être à support compact, et l'inégalité de Hardy correspondante ne vérifie alors ni (b), ni (c). Nous cherchons ici à obtenir de bonnes inégalités de Hardy : nous dirons qu'une inégalité de Hardy est *optimale* lorsqu'elle vérifie les trois propriétés (a), (b) et (c) vérifiées par (1) ci-dessus.

Cette Note comporte trois parties. Tout d'abord, nous présentons une méthode générale pour obtenir des inégalités de Hardy avec des poids explicites (Proposition 2.1). Notre méthode repose sur la simple observation que si  $u_0$  et  $u_1$  sont deux solutions positives de  $Pu = 0$ , alors pour  $\alpha \in (0, 1)$ ,  $u_\alpha := u_0^{1-\alpha} u_1^\alpha$  est une supersolution de  $P$ , et une solution de  $(P - W_\alpha)u_\alpha = 0$ , où  $W_\alpha$  est un poids positif explicite. La forme particulière de  $W_\alpha$  nous permet d'optimiser en  $\alpha$ . On obtient alors, pour  $\alpha = 1/2$ , un poids positif explicite (dépendant de  $u_0$  et  $u_1$ , ainsi que de leur gradient)  $W_{1/2}$  tel que l'inégalité de Hardy pour  $P$  de poids  $W_{1/2}$  soit vérifiée (voir (7)). Un exemple important de cette construction est le cas où  $u_0 = \mathbf{1}$  (donc,  $P\mathbf{1} = 0$ ), et  $u_1 = G = G(\cdot, 0)$ , la fonction de Green de  $P$  de pôle 0 : alors  $W_{1/2} = \frac{1}{4} |\nabla \log(G)|_A^2$ . Ce dernier exemple n'est pas nouveau, et apparaît déjà dans [4,5] et [7]; par contre, la méthode pour l'obtenir est, elle, nouvelle. En particulier, notre méthode permet d'obtenir des inégalités de Hardy généralisées lorsque  $P$  n'est pas symétrique.

Puis, nous montrons que la construction précédente donne dans certains cas des inégalités de Hardy optimales. Notre résultat principal traite le cas d'un domaine épointé  $\Omega^* := \Omega \setminus \{0\}$ , avec  $u_1 = G(\cdot, 0)$  et  $u_0 = u$  telle que  $\lim_{\infty} G/u = 0$ . On montre dans le Théorème 1 que l'inégalité de Hardy obtenue est *optimale*. L'ingrédient principal de la preuve est l'étude de l'ensemble des solutions de  $(P - \lambda W_{1/2})u = 0$ , selon les valeurs de  $\lambda$  : dû à la forme particulière de  $W_{1/2}$ , on montrera que pour  $\lambda < 1$ , l'équation a deux solutions indépendantes strictement positives, que pour  $\lambda = 1$ , il y a une solution strictement positive et une solution changeant de signe (de type logarithmique), et que pour  $\lambda > 1$  l'équation est oscillatoire. Pour illustrer ceci, considérons l'exemple de l'inégalité de Hardy classique (1). Si l'on cherche des solutions radiales  $u(x) = f(|x|)$  de l'équation  $(-\Delta - \lambda \frac{C_H}{|x|^2})u = 0$ , on voit que sur  $(0, \infty)$ ,  $f$  doit être solution de l'équation d'Euler

$$-f''(r) - \frac{n-1}{r} f'(r) - \lambda \frac{C_H}{r^2} f(r) = 0 \quad r \in (0, \infty). \tag{3}$$

Pour  $\lambda < 1$ , (3) a deux solutions positives linéairement indépendantes

$$f_{\pm}(r) = r^{(2-n)/2} (r^{(2-n)/2})^{\pm\sqrt{1-\lambda}}.$$

Pour  $\lambda = 1$ , deux solutions linéairement indépendantes de (3) sont  $f_+(r) = r^{(2-n)/2}$  (qui est une solution positive) et

$$f_-(r) = r^{(2-n)/2} \log(r^{2-n})$$

(qui change de signe). Finalement, pour  $\lambda > 1$ , (3) n'a pas de solution positive, mais des solutions oscillantes

$$f_+(r) = r^{(2-n)/2} \cos\left(\frac{\sqrt{\lambda-1}}{2} \log(r^{2-n})\right), \quad f_-(r) = r^{(2-n)/2} \sin\left(\frac{\sqrt{\lambda-1}}{2} \log(r^{2-n})\right).$$

Dans le cas général, l'observation cruciale est qu'il suffit – dans le cas où  $u = \mathbf{1}$  – de remplacer  $r^{2-n}$  par  $G$ , et les formules précédentes fournissent des solutions de  $P - \lambda W_{1/2}$  ! L'existence de la solution logarithmique entraîne la criticité de  $P - W_{1/2}$ , tandis que l'existence des solutions oscillantes implique l'optimalité de la constante à l'infini.

Enfin, notre méthode nous permet d'obtenir des inégalités de Hardy optimales dans d'autres cas que celui d'un espace épointé : nous considérons dans le Théorème 3.1 l'exemple du demi-espace  $\mathbb{R}_+^n$  pour  $P = -\Delta$  avec les fonctions harmoniques  $u_0(x) := x_n$ , et  $u_1(x) := x_n |x|^{-n}$ . L'inégalité de Hardy optimale obtenue est

$$\int_{\mathbb{R}_+^n} |\nabla u|^2 \geq \frac{n^2}{4} \int_{\mathbb{R}_+^n} \frac{u^2}{|x|^2} \, dx \quad \forall u \in C_0^\infty(\mathbb{R}_+^n). \tag{4}$$

Cette inégalité apparaît dans [6], où l'optimalité de la constante est aussi obtenue, mais la criticité de l'opérateur est un fait nouveau. Mentionnons aussi qu'un résultat analogue est vrai plus généralement dans le cas de domaines coniques (sous des hypothèses supplémentaires sur  $P$ ); ceci apparaîtra dans un article ultérieur.

### 1. Introduction

The classical Hardy’s inequality in  $\mathbb{R}^n \setminus \{0\}$  reads as

$$\int_{\mathbb{R}^n \setminus \{0\}} |\nabla u|^2 dx \geq C_H \int_{\mathbb{R}^n \setminus \{0\}} \frac{u^2(x)}{|x|^2} dx \quad \forall u \in C_0^\infty(\mathbb{R}^n \setminus \{0\}), \tag{5}$$

where  $C_H := (\frac{n-2}{2})^2$ . The inequality has the following important features:

- (a) The weight  $\frac{C_H}{|x|^2}$  is *critical*. In particular,  $C_H$  is the optimal constant.
- (b)  $C_H$  is also optimal for test functions supported in any neighborhood of either 0 or  $\infty$ .
- (c) The corresponding variational problem admits no minimizer.

There have been many generalizations of (5) to general domains and operators (see [3] and the references therein). Broadly speaking, given a symmetric, second-order, nonnegative elliptic operator  $P$  on a domain  $\Omega \subset \mathbb{R}^n$  and its associated quadratic form  $q$ , a *Hardy-type inequality* has the form

$$q(u, u) \geq \int_{\Omega} W u^2 dx \quad \forall u \in C_0^\infty(\Omega), \tag{6}$$

where  $W$  is a nonzero nonnegative potential on  $\Omega$ . The set of nonzero nonnegative potentials  $W$  satisfying (6) is nonempty (and huge) if and only if  $P$  is subcritical in  $\Omega$ . In the subcritical case, this set is a convex set  $\mathcal{W}$ , whose extremal points are the *critical potentials*. However, critical potentials can be compactly supported and for such potentials (6) features neither (b) nor (c). Indeed, our aim is to find generalizations of the classical Hardy inequality that would share all mentioned features. We shall thus say that a Hardy-type inequality is *optimal* if it satisfies all the three properties (a)–(c).

In this Note, we present a method for constructing potentials  $W$  such that Hardy-type inequalities hold. The method is based on the Agmon–Allegretto–Piepenbrink (AAP) theory and allows us to study properties (a), (b), (c) in a single framework. Furthermore, (6) naturally extends to the case of a nonsymmetric  $P$ . In the punctured domain  $\Omega^* := \Omega \setminus \{0\}$  the method gives an optimal Hardy-type inequality. The method is not limited to the setting of a punctured domain, and we illustrate this in the last paragraph by the example of the upper-half space  $\mathbb{R}_+^n$  for  $P = -\Delta$ .

Our Hardy-type inequalities have precursors in [1,4,5,7]. However, the generality of these inequalities (including the nonsymmetric case), the connection to the (AAP) theory, and the coherence in delivering (a), (b), (c) is novel. In particular, the criticality property (a) is new, independently of the method.

### 2. Construction of a Hardy weight

We assume that  $Pu = -\text{div}(A\nabla u) + \mathbf{b} \cdot \nabla u - \text{div}(\mathbf{c}u) + du$  is a second-order locally uniformly elliptic operator with real and locally regular coefficients defined on a domain  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 2$  (or on a noncompact manifold). For  $x \in \Omega$  we introduce the norm  $|\xi|_A^2 := \xi \cdot A(x)\xi$  defined on  $\mathbb{R}^n$ .

We first explain the framework. The (AAP) theorem (see [8]) states that the quadratic form  $q$  of a symmetric operator  $P$  is nonnegative on  $C_0^\infty(\Omega)$  if and only if the cone  $\mathcal{C}_P(\Omega)$  of positive solutions of the equation  $Pu = 0$  in  $\Omega$  is nonempty. Hence, for a general operator  $P$  and a nonnegative potential  $W$ , we define  $\lambda_0(P, W, \Omega)$ , the *generalized principal eigenvalue*, by

$$\lambda_0(P, W, \Omega) := \sup\{\lambda \geq 0 \mid \mathcal{C}_{P-\lambda W}(\Omega) \neq \emptyset\},$$

and we say that a *Hardy-type inequality* holds with weight  $W$  if  $\lambda_0(P, W, \Omega) \geq 1$ . In the symmetric case, this is equivalent to (6). In the nonsymmetric case it naturally extends (6). For the sake of brevity, we say that  $P - W$  is nonnegative in  $\Omega$ , and write  $P - W \geq 0$ , whenever  $\mathcal{C}_{P-W}(\Omega) \neq \emptyset$ .

Our construction of weights satisfying the Hardy inequality is as follows. We assume that  $P \geq 0$  is subcritical, and that one can find two noncolinear functions  $u_0$  and  $u_1$  in  $\mathcal{C}_P(\Omega)$ . It is well known that for  $\alpha \in (0, 1)$ ,  $u_\alpha := u_0^{1-\alpha}u_1^\alpha$  is a positive supersolution of  $P$ . In fact,

$$(P - 4\alpha(1 - \alpha)W(u_0, u_1))u_\alpha = 0, \quad \text{where } W(u_0, u_1) := \frac{1}{4} \left| \nabla \left( \log \frac{u_0}{u_1} \right) \right|_A^2. \tag{7}$$

Note that  $W$  does not depend on  $\alpha$ . Thus, optimizing on  $\alpha$ , we obtain:

**Proposition 2.1.** *The Hardy-type inequality  $P - W(u_0, u_1) \geq 0$  is valid in  $\Omega$ .*

### 3. Optimal Hardy inequalities

In the present section, we construct an *optimal* Hardy-type inequality in the punctured domain  $\Omega^*$ . We first explain the features of optimal weights, in particular we discuss the notion of criticality for a nonnegative operator  $P$  (for more details, see [8]). An operator  $P \geq 0$  is said to be *critical* in  $\Omega$  if there is a unique (up to a constant) positive supersolution  $\varphi$  of the equation  $Pu = 0$  in  $\Omega$ , otherwise,  $P$  is *subcritical* in  $\Omega$ . If  $P \not\geq 0$ , then  $P$  is said to be *supercritical* in  $\Omega$ . In the critical case, any positive supersolution  $\varphi$  is a solution, and  $\varphi$  is called the *ground state* of  $P$ . There are various equivalent characterizations of criticality and subcriticality. We will use two of them: subcriticality is equivalent to the existence of a positive minimal Green function, and is also equivalent to the statement that for every  $V \geq 0$  smooth, compactly supported, there exists  $\varepsilon > 0$  such that  $P - \varepsilon V \geq 0$ . We say that  $W \geq 0$  is a *critical weight* for  $P$  if  $P - W$  is critical in  $\Omega$ . Thus, if  $W$  is a critical weight for  $P$ , the inequality  $P - W \geq 0$  is (globally) optimal in the sense that for every potential  $V \geq W$ , the operator  $P - V$  is supercritical in  $\Omega$ .

Assume that  $P - W$  is critical in  $\Omega^*$ . We discuss the optimality of the constant near zero and near infinity of the generalized Hardy inequality  $P - W \geq 0$ . We define  $\lambda_\infty(P, W, \Omega)$  by

$$\lambda_\infty(P, W, \Omega) := \sup\{\lambda \geq 0 : \exists K \Subset \Omega \text{ s.t. } C_{P-\lambda W}(\Omega \setminus K) \neq \emptyset\}, \tag{8}$$

and say that the constant 1 is *optimal at infinity* if  $\lambda_\infty(P, W, \Omega) = 1$  (obviously,  $P - W \geq 0$  already gives us  $\lambda_\infty(P, W, \Omega) \geq 1$ ). Similarly, we define the optimality of 1 at zero.

Assume that  $P - W$  is critical in  $\Omega^*$ , let  $\varphi$  be its ground state, and  $\varphi^*$  be the ground state of the formal adjoint operator  $P^*$ . We say that  $P$  is *null-critical* in  $\Omega^*$  (respectively at infinity) with respect to  $W$ , if  $\varphi\varphi^* \notin L^1(\Omega^*, W \, dx)$  (resp.  $\varphi\varphi^* \notin L^1(\Omega \setminus K, W \, dx)$ , where  $K$  is a compact neighborhood of 0). Similarly, null-criticality at zero is defined. In the symmetric case, null-criticality in  $\Omega$  is equivalent to the nonexistence of a minimizer for the variational problem

$$\inf\left\{ \frac{q(u, u)}{\int_\Omega u^2 W \, dx} \mid u \in \mathcal{D}^{1,2}(\Omega), u \neq 0 \right\},$$

where  $\mathcal{D}^{1,2}(\Omega)$  is the completion of  $C_0^\infty(\Omega)$  with respect to the norm  $u \mapsto \sqrt{q(u, u)}$ .

Finally, we say that the weight  $W$  is *optimal* if the following hold: (a) the operator  $P - W$  is critical in  $\Omega$ ; (b) the constant 1 is optimal at zero and at infinity; (c) the operator  $P - W$  is null-critical at zero and at infinity. We emphasize that an optimal weight is of course *not unique*. In the symmetric case, the relation between optimal potentials and the bottom of the *essential* spectrum is demonstrated via (8) which is Persson's formula for the essential spectrum of the corresponding operator. Our main result is:

**Theorem 1.** *Let  $G(x) := G_P^\Omega(x, 0)$  be the Green function of a subcritical operator  $P$  with pole at  $0 \in \Omega$ . Assume that there exists  $u \in C_P(\Omega)$  such that*

$$\lim_{x \rightarrow \infty} \frac{G(x)}{u(x)} = 0 \tag{9}$$

(infinity being the ideal point in the one-point compactification of  $\Omega$ ). Then  $W(G, u)$  is an optimal weight for  $P$  in  $\Omega^*$ .

**Remark 1.** A positive solution  $u$  satisfying (9) always exists in the (quasi)symmetric case [2].

**Sketch of the proof of Theorem 1.** The proof hinges on an observation concerning the nature of special solutions of the equation  $(P - \lambda W)v = 0$ , when  $\lambda$  varies. To explain this, let us consider the particular case of the classical Hardy inequality in  $\mathbb{R}^n \setminus \{0\}$ ,  $n \geq 3$ . If we look at radial solutions  $u(x) = f(|x|)$  of  $(-\Delta - \lambda \frac{C_H}{|x|^2})u = 0$ , we see that  $f$  satisfies the celebrated Euler equation

$$-f''(r) - \frac{n-1}{r} f'(r) - \lambda \frac{C_H}{r^2} f(r) = 0 \quad r \in (0, \infty). \tag{10}$$

For  $\lambda < 1$ , (10) has two linearly independent positive solutions

$$f_\pm(r) = r^{(2-n)/2} (r^{(2-n)/2})^{\pm\sqrt{1-\lambda}}.$$

For  $\lambda = 1$ , the two linearly independent solutions of (10) are the positive solution  $f_+(r) = r^{(2-n)/2}$ , and the changing sign solution

$$f_-(r) = r^{(2-n)/2} \log(r^{2-n}).$$

Finally, for  $\lambda > 1$ , (10) does not have any positive solution, but two oscillating solutions

$$f_+(r) = r^{(2-n)/2} \cos\left(\frac{\sqrt{\lambda-1}}{2} \log(r^{2-n})\right), \quad f_-(r) = r^{(2-n)/2} \sin\left(\frac{\sqrt{\lambda-1}}{2} \log(r^{2-n})\right).$$

Notice that in the case of the Laplacian on  $\mathbb{R}^n$ ,  $r^{2-n}$  is (up to a constant) the Green function with pole 0, furthermore in this case one can take the solution  $u$  to be constant equal to 1. Strikingly enough, the general case follows along the same line. WLOG we may assume  $u = 1$ . Replacing in the above formulas  $r^{(n-2)}$  with  $G$ , we obtain solutions of the equation  $(P - \lambda W)u = 0$  in  $\Omega^*$ . The log solution is then responsible for the criticality of the operator  $P - W$ , whereas the oscillating solutions are responsible for the optimality of the constant 1 and the null criticality at infinity and at zero.  $\square$

To conclude this Note, we point out that an optimal weight can be constructed not only in punctured domains but in certain cases, also *globally* in  $\Omega$  (and also in the quasilinear case). For example, our construction applies (under some assumption on  $P$ ) in cone-like domains. We illustrate this in the particular case of  $P = -\Delta$  in the upper-half space  $\mathbb{R}_+^n$ . In a recent paper [6], it is proved that the following inequality holds with the best constant:

$$\int_{\mathbb{R}_+^n} |\nabla u|^2 dx \geq \frac{n^2}{4} \int_{\mathbb{R}_+^n} \frac{u^2}{|x|^2} dx \quad \forall u \in C_0^\infty(\mathbb{R}_+^n). \tag{11}$$

We can recover this inequality and strengthen it by applying our Proposition 2.1 with the harmonic functions  $u_0(x) := x_n$  and  $u_1(x) := x_n|x|^{-n}$ . In this case we consider two singular points 0 and  $\infty$ . The previous definitions extend in an obvious way to define an optimal weight on  $\mathbb{R}_+^n$ . So, the result of Theorem 1 holds true in this case:

**Theorem 3.1.** *The weight  $W(x) := n^2/(4|x|^2)$  is an optimal weight for  $-\Delta$  on  $\mathbb{R}_+^n$ .*

Note that except at zero,  $W$  is up to the boundary continuous in  $\mathbb{R}_+^n$ , which allows us to work on the space  $\overline{\mathbb{R}_+^n} \setminus \{0, \infty\}$ , and use a boundary Harnack principle which is valid for  $-\Delta - \lambda W$  in  $\overline{\mathbb{R}_+^n} \setminus \{0, \infty\}$ .

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