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Dynamical Systems

# Combinatorics and topology of the Robinson tiling

## *Combinatoire et topologie des pavages de Robinson*

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ABSTRACT

We study the space of all tilings which can be obtained using the Robinson tiles (this is a two-dimensional subshift of finite type). We prove that it has a unique minimal subshift, and describe it by means of a substitution. This description allows to compute its cohomology groups, and prove that it is a model set. *This article has an annex which was transmitted to the Académie des Sciences.*



RÉSUMÉ

Nous étudions l'espace de tous les pavages qui peuvent s'obtenir à partir des tuiles de Robinson (il s'agit d'un sous-décalage de type fini). Cet espace contient un unique sous-espace minimal, que nous décrivons par le biais d'une substitution. En conséquence, il est possible de calculer les groupes de cohomologie associés, et de montrer qu'il s'agit d'un pavage de coupe et projection. *Cet article a une annexe qui a été transmise à l'Académie des Sciences.*



### Version française abrégée

C'est en 1971 que Robinson introduit l'ensemble de tuiles qui porte son nom. Un « pavage de Robinson » est un pavage que l'on peut obtenir à partir des tuiles de la Fig. 1 (ainsi que leurs images par rotation et reflexion). Les pavages de Robinson doivent en outre respecter les règles suivantes : les tuiles doivent se rencontrer face-à-face, et les flèches doivent rencontrer des lignes ; par ailleurs, dans une colonne sur deux et une ligne sur deux, une tuile sur deux est de type (a) (voir Fig. 1), sans restriction a priori sur son orientation. Les tuiles de type (a) sont appelées des « carrefours ». Les tuiles de Robinson sont un exemple important d'ensemble de tuiles apériodiques : aucun pavage construit en suivant ces règles n'a de période. Cependant, peu de choses sont connues quant à la topologie de l'espace de pavages associé.

Formellement, un pavage est une décoration de $\mathbf{Z}^2$ : à chaque élément du réseau correspond une tuile dans une orientation donnée. Ainsi, un pavage est un élément de $\mathcal{A}^{\mathbf{Z}^2}$, où $\mathcal{A}$ est l'ensemble des tuiles de Robinson. On note $\Xi$ l'ensemble des pavages de Robinson. C'est un sous-décalage de $\mathcal{A}^{\mathbf{Z}^2}$, c'est-à-dire un sous-ensemble fermé (donc compact), et invariant sous l'action de $\mathbf{Z}^2$ par décalage (translation). Un point important est que cet espace est non vide, et ne contient aucune période (voir Fig. 4 pour un amas de taille $7 \times 7$).

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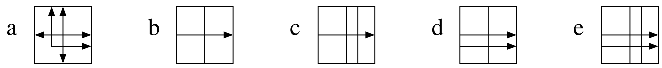

**Fig. 1.** The Robinson tiles. The tiles of the first type (a) are called *crosses*.

L'espace des pavages de Robinson n'est pas minimal, mais contient un unique sous-espace minimal, noté $\Xi_{\min}$. Le théorème principal de cet article s'énonce ainsi :

**Théorème 1.** *Il existe une substitution* (*voir ci-dessous*) $\tilde{\omega}$ *telle que l'espace de pavages associé à la substitution* $\Xi_{\tilde{\omega}}$ *est topologiquement conjugué à* $\Xi_{\min}$ *: il existe un homéomorphisme entre ces deux espaces qui commute aux actions.*

Dans cette Note, une substitution $\omega$ est une application qui associe a chaque tuile un carré de $2 \times 2$ tuiles. L'itération d'une substitution produit une suite d'amas de taille croissante, et par un passage à la limite adéquat, produit un pavage $T$. On appelle $\Xi_\omega$ le plus petit sous-décalage de $\mathcal{B}^{\mathbf{Z}^2}$ qui contient $T$. Sous certaines conditions qui sont satisfaites ici, l'espace $\Xi_\omega$ ne dépend que de $\omega$, et pas de $T$. Il est de plus minimal et sans période.

La preuve du théorème ci-dessus se fait en deux parties. D'abord, on exhibe une substitution $\omega$, décrite en Fig. 3. Notons $\Theta$ l'application « oubli » $\mathcal{A} \to \mathcal{A} \cup \{\text{blanche}\}$, qui est l'identité lorsque restreinte aux carrefours, et envoie toutes les autres tuiles sur la tuile blanche. Cette application s'étend aux pavages, et définit une application $\Xi_{\min} \to \Xi_\Theta$. Alors $\Xi_\omega$ et $\Xi_\Theta$ sont topologiquement conjugués. En d'autres termes, la substitution $\omega$ décrit entièrement l'emplacement des carrefours sur les pavages de Robinson. Cela dit, certains pavages de Robinson ne sont pas entièrement déterminés par la donnée de tous leurs carrefours. Ainsi, $\Xi_\omega$ n'est pas conjugué à $\Xi_{\min}$.

La seconde étape est donc de décorer la substitution $\omega$ en une substitution « augmentée », notée $\tilde{\omega}$, de sorte que les pavages obtenus par la nouvelle substitution décrivent l'emplacement des carrefours, mais aussi des autres tuiles qui composent les pavages de Robinson (les carrefours sont reliés par des lignes simples ou doubles). Le processus de décoration est illustré en partie par la Fig. 5, qui décrit comment encoder la position des lignes simples (ce sont les lignes fléchées des tuiles (b) et (c), et les lignes non fléchées des tuiles (b) et (d), voir Fig. 1).

La conjugaison entre $\Xi_{\min}$ et $\Xi_{\tilde{\omega}}$ est construite explicitement par des dérivations locales. On donne une manière de re-coder un pavage de $\Xi_{\min}$ en terme des tuiles de $\tilde{\omega}$, et réciproquement.

Ce résultat a deux applications notables. Tout d'abord, une telle description permet de calculer des invariants topologiques pour l'espace $\Xi_{\min}$, en utilisant les méthodes d'Anderson et Putnam [1]. Si on note $\Omega$ la suspension de $\Xi_{\min}$, on a :

$$\check{H}^2(\Omega) = \mathbf{Z}[1/4] \oplus \left(\mathbf{Z}[1/2]\right)^{10} \oplus \mathbf{Z}^8 \oplus \mathbf{Z}/4\mathbf{Z}; \qquad \check{H}^1(\Omega) = \left(\mathbf{Z}[1/2]\right)^2 \oplus \mathbf{Z}; \qquad \check{H}^0(\Omega) = \mathbf{Z}.$$

Par ailleurs, comme l'espace (minimal) des pavages de Robinson est substitutif, et que chaque element contient un sous-ensemble périodique de carrefours, on peut appliquer un théorème de Lee et Moody [4, théorème 3], qui implique que $\Xi_{\min}$ peut être décrit par la méthode de coupe et projection. En particulier, les spectres dynamique et de diffraction sont purement ponctuels.

Avant de conclure, notons qu'une substitution plus simple, mais avec recouvrement de tuiles, et décrivant les mêmes pavages de Robinson, a été découverte indépendemment par Joan Taylor (communication privée). Cette substitution permet aussi de calculer la cohomologie, et donne les mêmes résultats.

## 1. Introduction

In 1971, Robinson [5] introduced his aperiodic set of tiles, in order to build a two-dimensional subshift of finite type with no periodic orbit. See also [3] for a comprehensive presentation. It is still today an important example of a simple aperiodic set of tiles. However, little was known about the topology of the tiling space. The Robinson tiling space is the set of all tilings which can be built from the Robinson tiles as follows. Consider a set $\mathcal{A}$ of 28 symbols. These symbols are the tiles of Fig. 1, as well as their images under rotations and reflections. Then, consider the subshift of finite type $\Xi \subset \mathcal{A}^{\mathbf{Z}^2}$ defined by:

(i) adjacency relations: two neighbouring tiles should meet in such a way that arrowheads of a tile meet arrowtails of its neighbour;
(ii) alternating cross rule: for all $x \in \Xi$, there is an element $n \in \mathbf{Z}^2$, such that for all $i \in (2\mathbf{Z}) \oplus (2\mathbf{Z})$, $x_{n+i}$ is a *cross* (see Fig. 1). There may also be crosses at other positions.

Then, it is possible to show that $\Xi$ is not empty, and none of its elements has periods under the action of $\mathbf{Z}^2$ by translation. The key to this result is that any $x \in \Xi$ has a hierarchical structure: for all $n \in \mathbf{N}$, define an $n$-supercross as shown in Fig. 4 (a 2-supercross is given as an example, on the left). Supercrosses are admissible under the adjacency rules. Therefore, by taking an appropriate union, one can build an element of $\Xi$. Conversely, the matching rules force supercrosses to appear in any admissible tiling. This gives a hierarchical structure which allows to prove non-periodicity of any $x \in \Xi$.

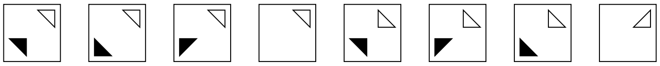

**Fig. 2.** The tiles of the substitution.

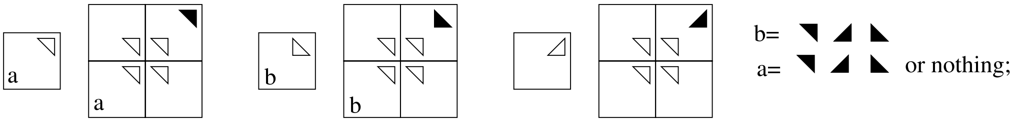


**Fig. 3.** The substitution.

An increasing union of $n$-supercrosses is called an *infinite order supertile*. Since there are several ways of including an $n$-supercross in an $(n+1)$-supercross, there exist many different infinite order supertiles. They need not cover the whole plane. Given $x \in \Xi$, we have the following **alternative**:

(i) either $x$ is made of only one infinite order supertile;
(ii) or $x$ contains several (actually 2 or 4) infinite order supertiles.

**Proposition 1.1.** *The subshift $\Xi$ contains a unique, minimal subspace, called $\Xi_{\text{min}}$. Any element of $\Xi$ which follows alternative* (i) *is in $\Xi_{\text{min}}$.*

To prove it, remark that any element of the tiling space contains $n$-supercrosses for all $n$. Therefore, elements which follow alternative (i) are accumulation points of any orbit of $\Xi$.

## 2. A substitution

We want to define a substitution map (see for example [1]) which describes the Robinson minimal tiling space. In our context, a substitution is a set of tiles $\mathcal{B}$, and a map $\omega : \mathcal{B} \to \mathcal{B}^{\{0,1\}\times\{0,1\}}$ which associates to every tile a $2\times 2$ patch of tiles. Then the substitution tiling space $\Xi_\omega$ is the set of all elements $x \in \mathcal{B}^{\mathbf{Z}^2}$ such that any patch $x_{[n,n+k)\times[m,m+l)}$ of any size $k \times l$ appears (up to translation) as a subpatch of $\omega^N(t)$ for some integer $N$ and some tile $t$. It is a closed, shift-invariant subset.

Consider the tiles given in Fig. 2, and the associated substitution given by Fig. 3. One can check that this substitution is *primitive*: for any two tiles $a$ and $b$, the tile $b$ is contained in the $n$-th substitution of $a$, for $n$ big enough. It is known that tiling spaces associated with primitive substitutions are *minimal*.

**Theorem 2.1.** *The tiling space $\Xi_\omega$ associated with the substitution above is a factor of $\Xi_{\text{min}}$, that is, there is an onto map $\phi : \Xi_{\text{min}} \to \Xi_\omega$, which commutes with the shift.*

We describe this factor more precisely in the next section. Then, we will prove the theorem. Let us mention without proof that this factor map is almost everywhere one-to-one. Moreover, we will explain below, by "decorating" the tiles and the substitution, how one can build a substitution $\tilde{\omega}$, whose tiling space is topologically conjugate to $\Xi_{\text{min}}$.

## 3. Local derivations

A map $\phi : \mathcal{A}^{\mathbf{Z}^2} \to \mathcal{B}^{\mathbf{Z}^2}$ is called a *local derivation* (see [2]) if there is a $C > 0$ such that for any $x$ and any position $(i, j)$, the tile $\phi(x)_{i,j}$ only depends on the translation class of the patch of radius $C$ in $x$ around $(i, j)$. Such a map is automatically continuous, and commutes with the shift.

Consider two minimal subshifts $\Xi_{\mathcal{A}} \subset \mathcal{A}^{\mathbf{Z}^2}$ and $\Xi_{\mathcal{B}} \subset \mathcal{B}^{\mathbf{Z}^2}$, and a local derivation $\phi$. If $x \in \Xi_{\mathcal{A}}$ and $\phi(x) \in \Xi_{\mathcal{B}}$, then by minimality, $\phi$ maps $\Xi_{\mathcal{A}}$ onto $\Xi_{\mathcal{B}}$, and therefore, it is a factor map. If there are two local derivations $\Xi_{\mathcal{A}} \to \Xi_{\mathcal{B}}$ and $\Xi_{\mathcal{B}} \to \Xi_{\mathcal{A}}$ which are inverse of each other, then the two subshifts are topologically conjugate. In order to prove Theorem 2.1, we need to define a local derivation $\Xi_{\text{min}} \to \mathcal{B}^{\mathbf{Z}^2}$, and then prove that at least one point of $\Xi_{\text{min}}$ is mapped to $\Xi_\omega$.

This local derivation is a composition of two maps $\phi = \phi_2 \circ \phi_1$. The map $\phi_1$ is a "forgetful" map, which only remembers the position and orientation of the crosses. The map $\phi_2$ breaks the tiles and recomposes them. An example of a patch together with its image by $\phi$ is given by Fig. 4. More precisely, $\phi_1$ is defined as follows (whether a cross is subject or not to the alternating cross rule is a local information):

– The image of a cross which is subject to the alternating cross rule is a tile decorated with empty triangles which remember the orientation of the cross.
– The image of any other cross is a tile decorated with solid black triangles, which remember the orientation of the cross.
– The image of any other tile is a blank (empty) tile.

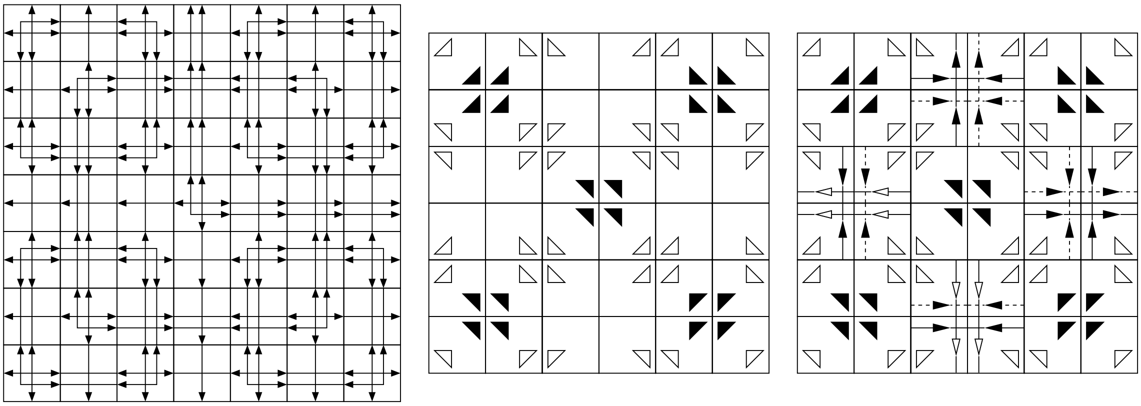

**Fig. 4.** On the left, a 2-supercross: it is made of four 1-supercrosses, and one cross in the middle. The middle cross is connected to lines which extend to the boundary of the supercross: the supercross behaves like a cross of bigger size. In the middle, the image of this supercross by the local derivation $\phi$ is shown. On the right, a fully decorated version of this supercross (see Section 4) is given.

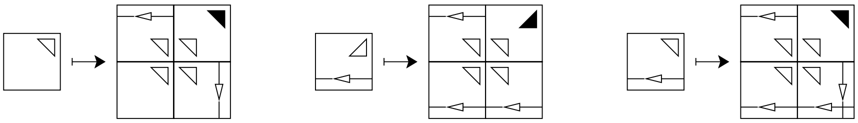

**Fig. 5.** Single lines are coded by white-arrowed lines. In the first substitution, arrows were added near two edges. If such a patch occurs in a tiling $y \in \Xi_\omega$, then any $x \in \Xi_{\min}$ which maps to $y$ will have a cross at the position of the black triangle. So, the white-arrowed lines in $y$ correspond to single lines in $x$. This creates new tiles. In the middle, the substitution of the lower-right tile of the first patch is shown, and on the right, the substitution of the lower-left tile of the second patch.

Remark that in a supercross, every row and every column contains at least one cross. Therefore, all the tiles in a supercross are entirely determined by the positions of the crosses. For this reason, this "forgetful" derivation is one-to-one when restricted to the set of Robinson tilings which follow alternative (i).

The map $\phi_2$ is defined by cutting all tiles obtained above in four, and recomposing them, so that the new tiles are now made of four pieces of four different previous tiles. The set of tiles which can be obtained is exactly $\mathcal{B}$, on which the substitution is defined. This derivation is of course invertible, by cutting tiles again and recomposing them (see Fig. 4). We call $\phi$ the composition of $\phi_1$ and $\phi_2$.

In order to prove that $\phi$ maps $\Xi_{\min}$ to $\Xi_\omega$, we need to iterate the substitution $\omega$ on a tile. Remark that for any tile $t$, $\omega(t)$ contains the image of a cross under the local derivation. By iteration, one proves that $\omega^n(t)$ contains an $(n-1)$-supercross (0-supercrosses being simply crosses). Therefore, the image under the local derivation of any Robinson tiling made of a single infinite order supercross is contained in $\Xi_\omega$. We conclude, using minimality, that $\Xi_\omega$ is a factor of $\Xi_{\min}$.

## 4. Decorating the substitution

The idea is now to decorate tiles of $\mathcal{B}$ to get a set of tiles $\tilde{\mathcal{B}}$, and a substitution $\tilde{\omega}$ on it. If we call $\psi$ the map $\tilde{\mathcal{B}} \to \mathcal{B}$ which forgets the decorations, we require that $\psi \circ \tilde{\omega} = \omega \circ \psi$ (so that $\omega$ is induced by $\tilde{\omega}$ on undecorated tiles). Then, $\psi$ induces a local derivation $\Xi_{\tilde{\omega}} \to \Xi_\omega$. The augmented substitution $\tilde{\omega}$ should be defined in such a way that the two spaces $\Xi_{\min}$ and $\Xi_{\tilde{\omega}}$—of which $\Xi_\omega$ is a factor—are conjugate.

Note that $\Xi_\omega$ and $\Xi_{\min}$ cannot be conjugate to each other: tilings containing more than one infinite order supertile in $\Xi_{\min}$ are mapped to tilings in $\Xi_\omega$ which may have several pre-images. Indeed, in such a tiling, there may be one row or one column in which there is no cross. The map $\phi$ forgets everything about such a line. In order to get a space which is conjugate to $\Xi_{\min}$, we need to keep track in $\tilde{\omega}$ of the lines which connect crosses (simple lines, double lines, together with arrows orienting them).

### *4.1. Keeping track of simple lines*

We describe how to produce decorations to keep track of simple lines (arrowed lines in tiles of type (b) and (c), and non-arrowed lines in tiles of type (b) and (d), see Fig. 1). Simple lines connect crosses which are "back-to-back". Therefore, we add decorations to the tiles in $\tilde{\mathcal{B}}$ in order to remember this fact (see Fig. 5). Next, we decide how to substitute a decorated tile. This creates new decorated tiles which are not yet in $\tilde{\mathcal{B}}$. Then, we decide how these decorated tiles substitute. We iterate this process until no new tile is created. Finally, we discard any tile which does not appear any longer in the eventual range of our decorated substitution.

In order to have a fully decorated substitution, we also need to keep track of double lines. See Fig. 4 (patch on the right) for an example of a fully decorated patch. The decorated substitution $\tilde{\omega}$ requires 208 different tiles, up to translation.

**Theorem 4.1.** *Call $\tilde{\omega}$ the substitution on decorated tiles. Then $\Xi_{\min}$ and $\Xi_{\tilde{\omega}}$ are topologically conjugate.*

To prove this, we found rules mapping any $2 \times 2$ patch of decorated tiles to a tile in $\mathcal{A}$. This gives a local derivation $\Xi_{\tilde{\omega}} \to \Xi_{\min}$. Conversely, there is a derivation from $2 \times 2$ Robinson patches to tiles in $\tilde{\mathcal{B}}$. The two derivations are inverses of each other. This was proved by checking that the image of any $3 \times 3$ patch of a Robinson tiling under the composition of the two derivations is the central tile of the original patch. The derivations are described in an appendix which was transmitted to the editor. The complete proof was assisted by a computer (the programs can be obtained upon request from the first author).

## 5. Applications

Since we now have a substitution $\tilde{\omega}$ on $\tilde{\mathcal{B}}$, it is possible to use the methods developed in [1] to compute the cohomology of the continuous hull of $\Xi_{\min}$. By continuous hull (or tiling space), we mean the suspension of the action of $\mathbf{Z}^2$ on $\Xi_{\min}$: if $\sigma^{(n,m)}(x)$ is the image of $x$ by the shift, define:

$$\Omega = \left(\Xi_{\min} \times \mathbf{R}^2\right)/\left\{\left(x, (t_1, t_2)\right) \sim \left(\sigma^{(n,m)}(x), (t_1 - n, t_2 - m)\right)\right\}.$$

Given a substitution, one can associate a finite CW-complex $\Gamma$, with 2-cells being (the interior of) the tiles. Then, the tiling space is homeomorphic to the inverse limit of $\Gamma$ under a map induced by the substitution. Using a computer program, it was possible to determine this complex (adjacencies, etc.), compute its Čech cohomology, and the map induced by the substitution on cohomology.

**Theorem 5.1.** *The cohomology groups of the Robinson minimal tiling space are*:

$$\mathrm{H}^2(\Omega) = \mathbf{Z}[1/4] \oplus \left(\mathbf{Z}[1/2]\right)^{10} \oplus \mathbf{Z}^8 \oplus \mathbf{Z}/4\mathbf{Z}; \qquad \mathrm{H}^1(\Omega) = \left(\mathbf{Z}[1/2]\right)^2 \oplus \mathbf{Z}; \qquad \mathrm{H}^0(\Omega) = \mathbf{Z}.$$

Another consequence of the fact that the Robinson tiling can be described by a (lattice) substitution is that it is a model set. As every element of the hull contains a lattice periodic subset of crosses, this follows from Theorem 3 of [4]. As a further consequence, it also implies that every tiling in the hull has pure point dynamical and diffraction spectrum.

Before concluding, let us mention that a somewhat simpler substitution, although with overlapping tiles, has been found independently by Joan Taylor (private communication). Also that substitution can be used to compute the cohomology, and gives the same results.

## Acknowledgements

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