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Persistence of noncompact normally hyperbolic invariant manifolds in bounded geometry

Persistance des variétés invariantes normalement hyperboliques non-compactes dans la géométrie bornée

Jaap Eldering

Mathematical Institute, Utrecht University, Budapestlaan 6, 3584 CD Utrecht, The Netherlands

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ABSTRACT

We prove a persistence result for noncompact normally hyperbolic invariant manifolds in Riemannian manifolds of bounded geometry. The bounded geometry of the ambient manifold is a crucial assumption in order to control the uniformity of all estimates throughout the proof.

RÉSUMÉ

Nous démontrons un résultat de persistance pour les variétés invariantes normalement hyperboliques non-compactes dans une variété riemannienne de géométrie bornée. Il est crucial d'assumer que la variété ambiante est de géométrie bornée pour contrôler l'uniformité des estimations tout au long de la preuve.

1. Introduction

Normally hyperbolic invariant manifolds (NHIMs for short) are used in many areas of dynamical systems, for example, in singular perturbation theory. It is well known that compact NHIMs are persistent under any C^1-small perturbation, see [4,6], while Sakamoto [7] and Bates, Lu, and Zeng [1] have extended this to noncompact NHIMs in Euclidean and Banach spaces respectively. Our result is an extension to a general noncompact setting in Riemannian manifolds. Bounded geometry is a crucial additional ingredient, needed to formulate the necessary uniformity conditions which allow to replace compactness by uniformity throughout the proof. Bounded geometry can be viewed as a uniformity condition on the ambient manifold and is automatically satisfied for Euclidean space.

2. Bounded geometry

We follow Eichhorn [2] to define bounded geometry. Recall that the injectivity radius $r_{\mathrm{inj}}(x)$ at a point $x \in Q$ is the maximum radius for which the exponential map at x is a diffeomorphism, and that normal coordinates are defined as the inverse map.

E-mail address: J.Eldering@uu.nl.

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Definition 2.1. We say that a complete, finite-dimensional Riemannian manifold (Q, g) has k-th order bounded geometry when

(i) the global injectivity radius $r_{\mathrm{inj}}(Q) = \inf_{x\in Q} r_{\mathrm{inj}}(x)$ is positive, $r_{\mathrm{inj}}(Q) > 0$;
(ii) the Riemannian curvature R and its covariant derivatives up to k-th order are uniformly bounded,

$$\forall 0 \leqslant i \leqslant k: \ \sup_{x\in Q} \left\| \nabla^i R(x) \right\| < \infty,$$

with operator norm of $\nabla^i R(x)$ viewed as a multilinear map on $\mathrm{T}_x Q$.

Note that both Euclidean space and compact smooth Riemannian manifolds have bounded geometry of any order k, i.e. $k = \infty$. Less trivial examples of bounded geometry are symmetric spaces or spaces constructed as products or as compactly glued connected sums of bounded geometry spaces.

It follows from Theorem 2.4 in [2] that a manifold of bounded geometry has an atlas of normal coordinate charts such that for some fixed $\delta > 0$ there is a normal coordinate chart defined on each ball $B(x;\delta)$, and moreover, the representation of the metric g in each chart is C^k-bounded, uniformly over all charts. We shall work with this preferred atlas and measure the C^k norm of functions in the following way:

Definition 2.2. Let X, Y be Riemannian manifolds of $k+1$-bounded geometry and $f \in C^k(X; Y)$. We say that f is of class C^k_b when there exist $\delta_X, \delta_Y > 0$ such that for each $x \in X$ we have $f(B(x;\delta_X)) \subset B(f(x);\delta_Y)$ and the representation

$$\tilde{f}_x = \exp^{-1}_{f(x)} \circ f \circ \exp_x : B(0;\delta_X) \subset \mathrm{T}_x X \to \mathrm{T}_y Y \tag{1}$$

in normal coordinates is of class C^k_b (i.e. C^k-bounded), and moreover, the associated C^k-norms of $\tilde{f}_x$ are bounded uniformly in $x \in X$.

This is a natural definition: $k+1$-bounded geometry implies that coordinate transition maps are uniformly C^k-bounded, hence this definition is equivalent to measuring the C^k-norm of f at x in any normal coordinate chart $B(x';\delta_X)$ containing x. Classes $C^k_{b,u}(X;Y)$ and $C^{k,\alpha}_{b,u}(X;Y)$ of uniformly (Hölder) continuous functions can be defined analogously when X, Y are of $k+2$-bounded geometry. These ideas can also be extended to classes $C^{k,\alpha}_b$ and $C^{k,\alpha}_{b,u}$ of vector fields and submanifolds. We shall allow submanifolds to be non-injectively immersed.

3. Results

We use the following definition of normal hyperbolicity. The flow is assumed complete for simplicity.

Definition 3.1. Let (Q, g) be a smooth Riemannian manifold, $\Phi^t \in C^{r\geqslant 1}$ a flow on Q, and let $M \in C^{r\geqslant 1}$ be a submanifold of Q. Then M is called a normally hyperbolic invariant manifold of the dynamical system $(\mathbb{R}, Q, \Phi)$ if all of the following conditions hold true:

(i) M is invariant, i.e. $\forall t \in \mathbb{R}: \Phi^t(M) = M$;
(ii) there exists a continuous splitting

$$\mathrm{T}_M Q = \mathrm{T}M \oplus E^+ \oplus E^- \tag{2}$$

of the tangent bundle $\mathrm{T}Q$ over M with globally bounded, continuous projections π_M, π_+, π_- and this splitting is invariant under the linearized flow $\mathrm{D}\Phi^t = \mathrm{D}\Phi^t_M \oplus \mathrm{D}\Phi^t_+ \oplus \mathrm{D}\Phi^t_-$;
(iii) there exist real numbers $\rho_- < -\rho_M \leqslant 0 \leqslant \rho_M < \rho_+$ and $C_M, C_+, C_- > 0$ such that the following exponential growth conditions hold on the various subbundles:

$$\begin{aligned}
&\forall t \in \mathbb{R},\ (m,x) \in \mathrm{T}M: && \left\| \mathrm{D}\Phi^t_M(m)x \right\| \leqslant C_M e^{\rho_M |t|} \|x\|, \\
&\forall t \leqslant 0,\ (m,x) \in E^+: && \left\| \mathrm{D}\Phi^t_+(m)x \right\| \leqslant C_+ e^{\rho_+ t} \|x\|, \\
&\forall t \geqslant 0,\ (m,x) \in E^-: && \left\| \mathrm{D}\Phi^t_-(m)x \right\| \leqslant C_- e^{\rho_- t} \|x\|.
\end{aligned} \tag{3}$$

This definition corresponds to 'eventual absolute normal hyperbolicity' in [6], and is slightly more restrictive than the 'relative normal hyperbolicity' definition in [6] that is also used in [4]. We say that M is an r-NHIM if the more general spectral gap condition

$$\rho_- < -r\rho_M \leqslant 0 \leqslant r\rho_M < \rho_+ \quad \text{with } r \geqslant 1 \tag{4}$$

on the growth exponents above is satisfied.

Theorem 3.2. *Let $k \geqslant 2$, $\alpha \in [0,1]$ and $r = k + \alpha$. Let (Q, g) be a smooth Riemannian manifold of bounded geometry and $v \in C^{k,\alpha}_{b,u}$ a vector field on Q. Let $M \in C^{k,\alpha}_{b,u}$ be a connected, complete submanifold of Q that is r-normally hyperbolic for the flow defined by v, with empty unstable bundle, i.e.* $\operatorname{rank}(E^+) = 0$.

Then for each sufficiently small $\eta > 0$ there exists a $\delta > 0$ such that for any vector field $\tilde{v} \in C^{k,\alpha}_{b,u}$ with $\|\tilde{v} - v\|_1 < \delta$, there is a unique submanifold $\tilde{M}$ in the η-neighborhood of M, such that $\tilde{M}$ is diffeomorphic to M and invariant under the flow defined by $\tilde{v}$. Moreover, $\tilde{M}$ is $C^{k,\alpha}_{b,u}$ and the distance between $\tilde{M}$ and M can be made arbitrarily small in C^{k-1}-norm by choosing $\|\tilde{v} - v\|_{k-1}$ sufficiently small.

Let us make some remarks on this result.

(i) The spectral gap condition (4) of r-normal hyperbolicity is essential to the proof. The $C^{k,\alpha}$ smoothness result is optimal. The minimum smoothness requirement $k \geqslant 2$ is a stronger assumption than $k \geqslant 1$ in the well-known compact case. This seems to be intrinsic to the noncompact case, cf. Hypothesis H2 in [1]. If the spectral gap condition only holds for some $1 \leqslant r < 2$, then we can still obtain a perturbed manifold $\tilde{M}$, but this manifold will generally not have better than C^r smoothness.

(ii) It should be possible to improve this result by lifting some of the technical restrictions. First of all, an unstable bundle E^+ can be added for full normal hyperbolicity. It should hold that the persistent manifold $\tilde{M}$ is an r-NHIM again.

(iii) Definition 3.1 could be relaxed to the more general definition of 'relative normal hyperbolicity' as used in [4,6,1]. This would require using the graph transform method; our Perron method proof seems tied to the current definition.

(iv) We only obtain a C^{k-1}-norm estimate for the perturbation distance of $\tilde{M}$ away from M, even though $\tilde{M} \in C^{k,\alpha}$ is preserved. It should be possible to improve this to the perturbation being $C^{k,\alpha}$-small when $\|\tilde{v} - v\|_{k,\alpha}$ is small.

By standard phase space extension techniques, we obtain the following results as a corollary:

Corollary 3.3. *Assume the setting of Theorem 3.2. If the vector field $\tilde{v}$ also depends on time, i.e. $\tilde{v} \in C^{k,\alpha}_{b,u}(\mathbb{R} \times Q)$, then there still exists a persistent manifold $\tilde{M} \in C^{k,\alpha}_{b,u}$, although it may be time-dependent. Similarly, if the vector field $\tilde{v}$ depends on an external parameter $p \in R^n$ and M is an r-NHIM for $p = 0$, then there exists a neighborhood $U \ni 0$ such that for each $p \in U$ we have a unique persistent manifold $\tilde{M}_p \in C^{k,\alpha}_{b,u}$ and $\tilde{M}_p$ depends $C^{k,\alpha}$ on p.*

4. Idea of the proof of Theorem 3.2

The following is only a rough sketch of the proof of Theorem 3.2, for a detailed exposition see [3].

We first reduce the problem to a trivial bundle $X \times Y$ where X is constructed as a manifold of bounded geometry of sufficiently high order (say $k+10$) that approximates M. Then we embed the normal bundle N of X into $X \times Y$ with $Y = \mathbb{R}^n$ for some n. A uniform tubular neighborhood of M can be modeled on N since M is the graph of a small function $h : X \to Y$. Additional normally hyperbolic dynamics can be added in the directions of Y complementary to N.

We apply a generalization of the Perron method based on ideas in [5]. Let $\tilde{v}_X$ and $\tilde{v}_Y(x, y) = A(x)y + f(x, y)$ denote the horizontal and vertical parts of the vector field $\tilde{v}$ respectively. If $(x(t), y(t))$ is a curve in $X \times Y$ with $y(t)$ uniformly small, then we denote by $\Phi_y(t, t_0, x_0)$ the flow of $\tilde{v}_X(\cdot, y(t))$ and by $\Psi_x(t, t_0)$ the linear flow of $A(x(t))$ on Y. A contraction map T is defined by $T(y, x_0) = T_Y(T_X(y, x_0), y)$ with

$$
\begin{aligned}
T_X(y, x_0)(t) &= \Phi_y(t, 0, x_0), \\
T_Y(x, y)(t) &= \int_{-\infty}^{t} \Psi_x(t, \tau) f\big(x(\tau), y(\tau)\big)\, \mathrm{d}\tau
\end{aligned}
\tag{5}
$$

mappings into appropriate spaces of curves in X, Y respectively. We finally recover the persistent manifold $\tilde{M}$ as the graph of the map $\tilde{h} : x_0 \mapsto \Theta(x_0)(0)$ where Θ denotes the fixed point of T as a function of the parameter $x_0 \in X$; the curve $\Theta(x_0)$ in Y is then evaluated at $t = 0$.

The $C^{k,\alpha}_{b,u}$ smoothness of Θ is proven inductively using ideas in [8] and the fiber contraction theorem. We introduce certain formal tangent bundles to work around the problem that the spaces of curves in X, Y are not (Banach) manifolds. We relate holonomy along closed loops in X to the curvature (which is bounded) to prove uniform continuity of the formal derivatives of T. Restricting the spaces of curves in X, Y to bounded time intervals turns these into Banach manifolds; this we use to finally recover true derivatives that lead to $\tilde{M} \in C^{k,\alpha}_{b,u}$.

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