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Differential Geometry

The Ricci continuity method for the complex Monge–Ampère equation, with applications to Kähler–Einstein edge metrics

La méthode de continuité de Ricci pour l'équation de Monge–Ampère complexe, avec des applications aux métriques de Kähler–Einstein conique le long d'arêtes

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ABSTRACT

In this Note we present a new continuity method and a priori $C^{2,\alpha}$ estimate for the degenerate complex Monge–Ampère equation. We then describe some applications of this method to the existence of Kähler–Einstein edge metrics, as conjectured by Tian and Donaldson.

RÉSUMÉ

Dans cette Note nous introduisons une nouvelle méthode de continuité et estimée $C^{2,\alpha}$ a priori, pour l'équation de Monge–Ampère complexe dégénérée. Nous présentons également quelques applications de cette méthode à l'existence de métriques de Kähler–Einstein ayant une structure conique le long d'arêtes, confirm des conjectures de Tian et de Donaldson.

Version française abrégée

Dans cette Note, nous introduisons une nouvelle méthode de continuité pour la construction de métriques de Kähler–Einstein singulières sur des variétés kählériennes compactes. Nous montrons que cette méthode est parfaitement adaptée pour la construction de métriques qui ont une structure conique le long d'arêtes (une arête sera pour nous un diviseur). L'existence de telles métriques a plusieurs applications que nous discuterons ailleurs.

Soit $D = D_1 + \cdots + D_k \subset M$ un diviseur à croisements normaux simples (SNC), où chaque D_i est une composante lisse. Soit s_i une section holomorphe du fibré en droites associé à D_i, avec une métrique hermitienne lisse h_i. Pour $\beta = (\beta_1, \ldots, \beta_k) \in (0, 1]^k$, soit $\omega_\beta := \omega_0 + \epsilon \sum_{i=1}^k \sqrt{-1}\partial\bar{\partial}(|s_i|^2_{h_i})^{\beta_i}$ la métrique à structure conique modèle d'angle $2\pi\beta_i$ le long de D_i.

Définition 0.1. Un courant de Kähler ω est un courant de Kähler–Einstein (KE), à structure conique le long d'une arête, de courbure de Ricci μ et angle conique $2\pi\beta_i$ le long de D_i, s'il existe $C > 0$ tel que $C^{-1}\omega_\beta \leqslant \omega \leqslant C\omega_\beta$ et si $\operatorname{Ric}\omega - \sum_i (1 - \beta_i)[D_i] = \mu\omega$, où $[D]$ est le courant d'intégration associé à D.

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Tian [15] a conjecturé que de telles métriques KE existent toujours quand $\mu < 0$, si l'on suppose que $c_1(M) - \sum_i (1-\beta_i) c_1(L_{D_i}) < 0$. Récemment, Donaldson [7] a conjecturé l'existence de telles métriques avec $0 < \mu = \beta \ll 1$ quand M est Fano et D est un diviseur anticanonique lisse. Le résultat suivant confirme ces conjectures de Tian et Donaldson. Dans le cas positif il faut utiliser une généralisation de la fonctionnelle de Mabuchi adaptée aux métriques à singularités coniques le long d'arêtes.

Théorème 0.2. *Soit $\Omega \in H^2(M,\mathbb{R})$ une classe de Kähler, $\mu \leqslant 0$, $\beta \in (0,1]^k$, avec $c_1(M) - \sum_i (1-\beta_i) c_1(L_{D_i}) = \mu\Omega$. Alors, il existe une métrique de Kähler–Einstein dont la courbure de Ricci μ représentant la classe Ω avec un angle $2\pi\beta_i$ le long de D_i. Dans le cas où $\mu > 0$, on dispose d'un résultat semblable si l'on suppose que la fonctionnelle adaptée de Mabuchi est propre. De plus, cette métrique et son tenseur de courbure admettent des développements asymptotiques polyhomogènes près de D.*

1. Main results and applications

Our goal in this Note is to describe a new continuity method for the construction of Kähler–Einstein metrics on compact Kähler manifolds, to show that it is very well suited for proving the existence of such metrics which are bent at an angle along a divisor, and finally to describe some uses for these singular metrics. (To be accurate, this continuity method was first introduced in the smooth setting by the second named author [14], and more recently adapted in [9] to the singular setting, and plays a similarly important role in the forthcoming paper [12].) It is well known that the Kähler–Einstein equation $\operatorname{Ric}\omega_\varphi = \mu\omega_\varphi$ (for some $\mu \in \mathbb{R}$) is equivalent to the complex Monge–Ampère equation

$$\omega_\varphi^n = \omega^n e^{f_\omega - \mu\varphi}, \tag{1}$$

where f_ω is defined by $\sqrt{-1}\partial\bar\partial f_\omega = \operatorname{Ric}\omega - \mu\omega$ and normalized by the condition $\int_M (e^{f_\omega} - 1)\omega^n = 0$.

When $\mu = 0$, this is known as the Calabi–Yau equation and the standard approach to solving it uses Calabi's continuity path [5]:

$$\omega_{\varphi(t)}^n = \omega^n e^{tf_\omega + c_t}, \quad t \in [0,1], \text{ where } e^{c_t} := \int_M \omega^n e^{f_\omega} \Big/ \int_M \omega^n e^{tf_\omega} \in \mathbb{R}_+;$$

the corresponding continuity path for $\mu < 0$ is $\omega_{\varphi(t)}^n = \omega^n e^{tf_\omega - \mu\varphi(t)}$, $t \in [0,1]$. When $\mu > 0$, following Aubin [2], the standard approach is to concatenate Calabi's path with $\omega_{\varphi(t)}^n = \omega^n e^{f_\omega - (t-1)\varphi(t)}$, $t \in [1, \mu+1]$. A key part of the analysis is to prove the a priori estimate $\|\varphi(t)\|_{C^{2,\alpha}(M,\omega)} \leqslant C$, where C depends on n, $\int_M \omega^n$, $\|\varphi(t)\|_{C^0(M)}$, and $\min_M \operatorname{Bisec}\omega$, and also on other quantities related to ω but that will always be bounded in our problems (such as the Sobolev and Poincaré constants of ω). Here $\min_M \operatorname{Bisec}\omega := \min_{\eta,\nu} \min_M \operatorname{Bisec}_\omega(\eta,\nu)$, where η, ν range over all unit vectors in $T^{1,0}M$, and $\operatorname{Bisec}_\omega(\eta,\nu) := R(\eta,\bar\eta,\nu,\bar\nu)$ is the bisectional curvature associated to the curvature tensor R of (M,ω). This estimate was proved by Aubin [1] and Yau [18] and was used by them to solve the complex Monge–Ampère equation (1) when $\mu < 0$ and $\mu \leqslant 0$, respectively, and by Tian [16] when $\mu > 0$ under a necessary and sufficient assumption on the Mabuchi K-energy functional [10].

We announce here a new continuity method for the *degenerate* complex Monge–Ampère equation, the *Ricci continuity method*, that has the advantage that it depends on an *upper* bound for the bisectional curvature rather than a lower one. This provides a new and unified proof of existence of solutions for (1) when M is smooth for all μ (and this observation goes back to [14]), but as we explain below, its real power lies in the fact that it is applicable in a range of interesting problems with singular background geometries. This Ricci continuity path is given by

$$\omega_{\varphi(s)}^n = \omega^n e^{f_\omega - s\varphi(s)}, \quad s \in (-\infty, \mu], \tag{2}$$

regardless of the sign of μ. This was already introduced in [14, p. 1533] in the context of Ricci iteration; indeed, for each fixed s, Eq. (2) is precisely the backward Euler discretization of the volume normalized Ricci flow $\frac{\partial\omega(t)}{\partial t} = -\operatorname{Ric}\omega(t) + \mu\omega(t)$ at time $\mu - \frac{1}{s}$.

Theorem 1.1. *Let $\varphi(s) \in C^4(M)$ be a solution of (2), with $s \geqslant S > -\infty$. There exist constants $C, \alpha > 0$ depending only on M, n, $\int_M \omega^n$, $\|\varphi(s)\|_{C^0(M)}$, S and $\max_M \operatorname{Bisec}\omega$, such that $\|\varphi(s)\|_{C^{2,\alpha}(M,\omega)} \leqslant C$.*

As noted above, this result gives a new proof of the Calabi–Yau theorem, but our main application of this result is to the construction of Kähler–Einstein metrics with an edge singularity along a divisor. In that setting, the Aubin–Yau estimates do not apply in the most interesting cases.

We begin with some definitions. Let $D = D_1 + \cdots + D_k \subset M$ be a divisor with simple normal crossings (SNC), where each D_i is a smooth component. Let s_i be a holomorphic section of the line bundle associated to D_i (and vanishing precisely along D_i), and h_i a smooth hermitian metric on this line bundle. Fix $\beta = (\beta_1,\ldots,\beta_k) \in (0,1]^k$, and set $\omega_\beta := \omega_0 + \epsilon\sum_{i=1}^k \sqrt{-1}\partial\bar\partial(|s_i|_{h_i}^2)^{\beta_i}$. We refer to ω_β as the model edge form with angle $2\pi\beta_i$ along D_i. It is easy to check that for $0 < \epsilon \ll 1$, ω_β is a Kähler current.

Definition 1.2. A Kähler current ω is called a Kähler–Einstein edge (KEE) current with Ricci curvature μ and cone angle $2\pi\beta_i$ along each D_i if $C^{-1}\omega_\beta \leqslant \omega \leqslant C\omega_\beta$ for some $C > 0$, and in addition, $\operatorname{Ric}\omega - \sum_i(1-\beta_i)[D_i] = \mu\omega$, where $[D]$ is the current associated to integration along D.

Tian [15] conjectured in 1994 that KEE metrics always exist when $\mu < 0$ assuming that $c_1(M) - \sum_i(1-\beta_i)c_1(L_{D_i}) < 0$. More recently, Donaldson [7] conjectured the existence of KEE metrics with $0 < \mu = \beta \ll 1$ when M is Fano and D is a smooth (i.e., $k = 1$) anticanonical divisor. The following result confirms these conjectures of Tian and Donaldson. The statement in the positive case uses a generalization of Mabuchi's K-energy to this setting.

Theorem 1.3. *Let $\Omega \in H^2(M,\mathbb{R})$ be a Kähler class, $\mu \leqslant 0$, $\beta \in (0,1]^k$, with $c_1(M) - \sum_i(1-\beta_i)c_1(L_{D_i}) = \mu\Omega$. Then there exists a KEE metric with Ricci curvature μ representing the class Ω with angle $2\pi\beta_i$ along each D_i. The same conclusion also holds when $\mu > 0$ provided the twisted Mabuchi K-energy is proper. Moreover, the KEE metric and its curvature tensor admit a complete polyhomogeneous asymptotic expansion near D.*

A proof of the special case of Theorem 1.3 when D is smooth appears in [9]; when D is smooth, but also $\beta \in (0,1/2]$ (this is the simpler "orbifold regime") and $\mu = 0$ the existence part of this theorem was also obtained by Brendle [4] using the Aubin–Yau estimate and Donaldson's work [8]. Finally, in the special case of Theorem 1.3 when $\mu \leqslant 0$ and $\beta_i \in (0,1/2]$ for all i, the existence part was obtained by an approximation technique by Campana–Guenancia–Păun [6]. Both [6] and [4] appeared simultaneously with [9]. However, if any $\beta_i > 1/2$, then by [9] the reference form ω_β no longer has bisectional curvature bounded from below, which underlines the importance of Theorem 1.1. The regularity part of this theorem (existence of polyhomogeneous expansion) turns out to play a crucial role in connecting the openness and closedness parts of the continuity argument. In addition, the linear estimates in [9] and [12] are independent of and more detailed than those in [8].

We now give some examples where this theorem may be applied. Recall that M is called a minimal manifold if the canonical bundle K_M is nef, i.e., $K_M \cdot C \geqslant 0$ for every holomorphic curve $C \subset M$. In particular, any Kähler manifold with $c_1(M) < 0$ is minimal. By the Aubin–Calabi–Yau theorem, when $c_1(M) < 0$ there is a smooth KE metric with negative Ricci curvature. Theorem 1.3 gives the following generalization:

Corollary 1.4. *Assume that M is a minimal projective manifold. Then M admits a KEE metric of negative Ricci curvature, and with any angle in $(0, 2\pi)$.*

Proof. By Bertini's theorem, M admits many smooth ample divisors. For any such divisor D, Kleiman's criterion gives that $c_1(M) - (1-\beta)c_1(L_D) < 0$ for any $\beta \in (0,1)$, so we may apply Theorem 1.3.

Tian [15] observed that such bendings yield a generalization of the Bogomolov–Miyaoka–Yau inequality, and indeed using Theorem 1.3 this can be carried through.

As for the case $\mu > 0$, we first state the following corollary to Theorem 1.3:

Corollary 1.5. *Let M be a Fano manifold and assume D is a smooth anticanonical divisor in M. Then there exists some $\beta_0 > 0$ such that for all $\beta \in (0, \beta_0)$ there exists a KEE metric with Ricci curvature β and angle $2\pi\beta$ along D.*

In related work [3], Berman proves the existence of a weak solution $\varphi \in L^\infty$ to (1), but gives neither $C^{2,\alpha}$ a priori estimates nor higher regularity near D. However, our proof of Corollary 1.5 relies on Berman's observation that when $0 < \mu = \beta \ll 1$ the twisted K-energy is proper, so that Theorem 1.3 applies.

It is well known that not all Fano manifolds admit a KE metric and it is a subtle and deep question to determine which ones do [16,7]. Corollary 1.5 implies that all Fano manifolds with a smooth anticanonical divisor admit a KEE metric with positive Ricci curvature. Such a divisor exists for many known examples, and, by results of Shokurov, this is always the case for Fano 3-folds. However, this is open in general (cf. the "Elephant conjecture" of Iskovskikh and Reid).

Even without the assumption on the existence of a smooth or SNC anticanonical divisor, our results imply that Fano manifolds always admit KEE metrics with zero or negative Ricci curvature. For example,

Corollary 1.6. $\mathbb{CP}^2$ *admits KEE metrics of constant positive, zero, and negative Ricci curvature.*

Allowing an edge along D acts as a sort of boundary condition for the problem and provides just enough flexibility to obtain a more robust existence theory, but is still sufficiently constrained that we see interesting rigidity phenomena.

This entire set of questions about the existence and behavior of KEE metrics as $2\pi\beta \nearrow 2\pi$, is reminiscent of, and indeed partially inspired by, a set of questions initiated by Thurston concerning hyperbolic 'conifolds', which are 3-dimensional hyperbolic pseudomanifolds bent along a (possibly singular) geodesic network. This theory is now rather well understood, see [13] for the newest progress. Passing to this limit in the KEE case when $\mu > 0$ is a key motivation for Donaldson's conjecture [7], though many difficult geometric and analytic problems remain.

2. Outline of the proofs

The proof of Theorem 1.1 appears in [9]. The key component is the use of a geometric inequality which can be traced back to Chern and Lu to obtain the Laplacian estimate, together with the facts that along (2) the Ricci curvature is bounded below by s and the twisted K-energy is monotonically decreasing, [14]. We turn to the proof of Theorem 1.3. A complete proof when D is smooth ($k=1$) can be found in [9], and the general case will appear in [12]. We sketch here some of the extra ideas needed when $k>1$.

We use the continuity path (2). The two main issues to address are the a priori estimates for the closedness part of the argument, and the tools from linear analysis needed to obtain openness along the path. The third issue is the existence of a solution when $s \ll -1$. It involves a perturbation argument similar to the one by Wu [17], but more subtle, since it has to be independent of any curvature assumptions.

The standard Moser iteration argument can be adapted to this singular setting to give the a priori C^0 estimate. Thus, following [18], $\|\varphi(s)\|_{C^0(M)}$ is bounded, for $S \leqslant s \leqslant 0$, in terms of the Poincaré and Sobolev constants of ω_β and the C^0 norm of f_ω, and these quantities are in turn controlled. More generally, adapting estimates of Tian [16] to the edge setting, we show that $\|\varphi(s)\|_{C^0(M)}$ is bounded for $s \in [0,\mu]$ if one assumes properness of the twisted Mabuchi K-energy. To carry this through we show that the edge geometries along the continuity path satisfy a uniform doubling measure property, which can be shown to imply a uniform Sobolev and Poincaré inequalities.

To estimate higher derivatives, we extend Theorem 1.1 to the edge setting. This requires some discussion of function spaces. Let $C_e^{k,\alpha}$ denote the 'edge' Hölder space, which is defined as the usual geometric $C^{k,\alpha}$ spaces with respect to the complete metric that near D is of the form $\omega_\beta/(|s_1|_{h_1}\cdots|s_k|_{h_k})^2$. Associated to this space is the Hölder–Friedrichs domain

$$\mathcal{D}_e^{0,\alpha} = \left\{u \in C_e^{2,\alpha}\colon\ \Delta_{\omega_\beta} u \in C_e^{0,\alpha}\right\}.$$

This comes equipped with a norm on $2+\alpha$ derivatives of u, albeit with certain loss of control near D. There is now a version of Theorem 1.1 that gives an a priori estimate on $\|\varphi(s)\|_{\mathcal{D}_e^{0,\alpha}}$ that depends on the same quantities as in that theorem. The following lemma then suffices for concluding the closedness:

Lemma 2.1. *The bisectional curvature of ω_β is bounded from above on $M \setminus D$.*

This upper bound is easy when all $\beta_i \leqslant 1/2$ since the entire curvature tensor is bounded then. The proof in the general case requires a lengthy computation, generalizing the one in [9] obtained by the second named author and C. Li when D is smooth, but is even more delicate near the intersection of the divisors.

Finally we discuss openness. The main issue is to give a precise description of regularity of elements of $\mathcal{D}_e^{0,\alpha}$. Since the Laplacian Δ_{ω_β} associated to ω_β is singular along D, it is rarely true that Δ_{ω_β} is bounded if we only know that $u \in C_e^{2,\alpha}$, so the property of lying in $\mathcal{D}_e^{0,\alpha}$ requires that u have some special regularity properties near D. The first result is as follows. Let G denote the Green operator for Δ_{ω_β} and K its nullspace in $L^2(M,\omega_\beta^n)$. We prove that $\mathcal{D}_e^{0,\alpha} := G(\mathcal{C}_e^{0,\alpha}) \oplus K = \{u = Gf\colon\ f \in \mathcal{C}_e^{0,\alpha}\} \oplus K$.

To use this effectively for the nonlinear problem, it is necessary to understand which 'Riesz potential operators', i.e. compositions of the form $\nabla \circ G$ or $\nabla^2 \circ G$ are bounded on $C_e^{0,\alpha}$. A very important insight used by Donaldson [8] is that since Δ_{ω_β} involves only mixed complex derivatives $\partial^2_{z_i\bar z_j}$, it is only necessary to understand the compositions of these with G, and he proves that $\partial^2_{z_i\bar z_j} \circ G$ is bounded. However, he uses Hölder spaces based on the incomplete metric ω_β, so his estimates are not strictly comparable to ours. Indeed, the edge Hölder spaces have several advantages. For example, in the edge space setting we can pass directly from Laplacian to $C^{2,\alpha}$ a priori estimates using a simple rescaling argument and quoting the standard Evans–Krylov interior estimates. What makes this fit into the overall scheme of the proof is the regularity result, that solutions of the complex Monge–Ampère equation are necessarily polyhomogeneous, and hence lie in the Hölder–Friedrichs spaces associated to ω_β too.

The tool used in [9] to prove linear mapping properties and regularity results for the simple edge case, i.e., $k=1$, is the calculus of pseudodifferential edge operators [11]. The analysis developed in [8] provides an easier path to some of the results in [11], but is perhaps not strong enough to prove all of the facts needed in the full existence proof. Unfortunately an equally strong theory of pseudodifferential operators associated to crossing edge singularities had not been developed yet, so one of the tasks in [12] is to develop such a theory. There are a number of simplifying features in this complex geometric setting to make this more tractable than the development of a full 'iterated edge' calculus, but unfortunately, analysis on a singular space with multi-level stratification is of necessity rather complicated. We conclude this Note by briefly describing some aspects of this calculus for the special case $k=2$, which contains all the general ideas but is less combinatorially complicated.

The first step is to perform a *real* blowup of M around D. When $k=1$ this is the space $\widetilde{M}$ obtained by replacing the points of D by their spherical normal bundles, which are just circles. Thus $\widetilde{M}$ is a manifold with boundary, where $\partial\widetilde{M}$ is the total space of a fibration over D with S^1 fibers, and the distance function r to D is a boundary defining function. When $k=2$, so $D = D_1 \cup D_2$, the normal bundles of these two components are disjoint away from 0, so we can blow up each D_i

as above independently of one another. In this case, the resulting space $\widetilde{M}$ is a manifold with corners up to codimension two; the two distance functions r_j, $j = 1, 2$, to D_j are now defining functions for the two boundary hypersurfaces $\partial_j \widetilde{M}$; each $\partial_j \widetilde{M}$ is again an S^1 bundle over $D_j \setminus (D_1 \cap D_2)$, while the corner $\partial_{12}\widetilde{M}$ is an $S^1 \times S^1$ bundle over $D_1 \cap D_2$.

The operator Δ_{ω_β} is singular at the boundary and corners of $\widetilde{M}$, so following original ideas of Melrose (in a different setting) we define a space of degenerate pseudodifferential operators adapted to this geometry to analyze the Green function G of Δ_{ω_β}. A *crossing edge pseudodifferential operator* A on $\widetilde{M}$ is characterized by the regularity of its Schwartz kernel, which is a distribution on $\widetilde{M}^2$. To specify this regularity efficiently, we define a new *crossing edge double space* $\widetilde{M}^2_{\mathrm{ce}}$, itself a real blowup of $\widetilde{M}^2$ along certain submanifolds of the boundary and corners. When $k = 1$ it suffices to blow up the diagonal of $(\partial\widetilde{M})^2$. When $k = 2$ the analogous submanifolds for each D_j are no longer transverse, so we first blow up their intersections. Thus $\widetilde{M}^2_{\mathrm{ce}}$ is obtained from $\widetilde{M}$ by an iterated blowup, first of this intersection of diagonals of the boundary faces and then of these diagonals. There is a blowdown map $b : \widetilde{M}^2_{\mathrm{ce}} \to \widetilde{M}^2$.

We say that an operator A is a pseudodifferential crossing edge operator if its Schwartz kernel is the pushforward under the blowdown map b of a distribution on $\widetilde{M}^2_{\mathrm{ce}}$ which has a standard pseudodifferential singularity along the diagonal of this product and polyhomogeneous expansions along each of the boundary hypersurfaces and product type expansions at the corners of $\widetilde{M}^2_{\mathrm{ce}}$. These various blowups separate the different regions where Δ_β and its Green function G have different types of expansions. The fact that we are working with distributions with polyhomogeneous expansions allows us to carry out iterative parametrix constructions in this calculus through a generalization of the one for the simple edge case in [11]. The upshot of this entire development is:

Theorem 2.2. *Let Δ be the Laplace operator associated to the reference metric ω_β, or to any solution of the complex Monge–Ampère operator along the continuity path* (2)*. Then the Green function G corresponding to the Friedrichs extension of Δ is an element of the pseudodifferential crossing edge calculus.*

In the interest of space, we do not introduce the notation necessary to give a detailed statement of this result. The paper [9] contains a substantial amount of expository material about the corresponding statement in the simple edge case, and the result in this more general setting is in the same spirit, although involving many more technicalities.

The proof of this theorem rests on an elliptic parametrix construction which takes into account the degenerate structure of Δ at each of the boundary faces. Having determined the structure of G, it is then easy to show that the Riesz potentials $\partial^2_{z_i\overline{z_j}} \circ G$ are again edge operators of the same type, so their boundedness on the edge Hölder spaces can be deduced from general boundedness theorems for general pseudodifferential crossing edge operators.

These pseudodifferential calculi are, at heart, simply effective formalisms to pass from the structure of the Green function of the Laplacian for the model problem, i.e., on the product space $C_{\beta_1}(S^1) \times \cdots \times C_{\beta_k}(S^1) \times \mathbb{R}^{2n-2k}$, to that of the Green function for the Laplacian of a crossing edge metric on M. However, this passage should not be regarded as routine since it is important in all the applications described above to know the precise polyhomogeneous structure of G, which require a much finer analysis than if we just ask for simpler bounds on this operator.

References

[1] T. Aubin, Équations du type Monge–Ampère sur les variétés kählériennes compactes, Bull. Sci. Math. 102 (1978) 63–95.

[2] T. Aubin, Réduction du cas positif de l'équation de Monge–Ampère sur les variétés kählériennes compactes à la démonstration d'une inégalité, J. Funct. Anal. 57 (1984) 143–153.

[3] R.J. Berman, A thermodynamical formalism for Monge–Ampère equations, Moser–Trudinger inequalities and Kähler–Einstein metrics, preprint, arXiv:1011.3976.

[4] S. Brendle, Ricci flat Kähler metrics with edge singularities, preprint, arXiv:1103.5454v2.

[5] E. Calabi, On Kähler manifolds with vanishing canonical class, in: R.H. Fox (Ed.), Algebraic Geometry and Topology. A Symposium in Honor of S. Lefschetz, Princeton University Press, 1957, pp. 78–89.

[6] F. Campana, H. Guenancia, M. Păun, Metrics with cone singularities along normal crossing divisors and holomorphic tensor fields, preprint, arXiv:1104.4879v2.

[7] S.K. Donaldson, Discussion of the Kähler–Einstein problem, 2009, preprint, available at: http://www2.imperial.ac.uk/~skdona/KENOTES.PDF.

[8] S.K. Donaldson, Kähler metrics with cone singularities along a divisor, preprint, arXiv:1102.1196v2.

[9] T. Jeffres, R. Mazzeo, Y.A. Rubinstein, Kähler–Einstein metrics with edge singularities (with an appendix by C. Li and Y.A. Rubinstein), preprint, arXiv:1105.5216.

[10] T. Mabuchi, K-energy maps integrating Futaki invariants, Tôhoku Math. J. 38 (1986) 575–593.

[11] R. Mazzeo, Elliptic theory of differential edge operators, I, Comm. PDE 16 (1991) 1616–1664.

[12] R. Mazzeo, Y.A. Rubinstein, Kähler–Einstein metrics with crossing edge singularities, in preparation.

[13] G. Montcouquiol, H. Weiss, Complex twist flows on surface group representations and the local shape of the deformation space of hyperbolic cone 3-manifolds, preprint, arXiv:1006.5582, 2010.

[14] Y.A. Rubinstein, Some discretizations of geometric evolution equations and the Ricci iteration on the space of Kähler metrics, Adv. Math. 218 (2008) 1526–1565.

[15] G. Tian, Kähler–Einstein metrics on algebraic manifolds, in: Transcendental Methods in Algebraic Geometry, Cetraro, 1994, in: Lecture Notes in Math., vol. 1646, 1996, pp. 143–185.

[16] G. Tian, Kähler–Einstein metrics with positive scalar curvature, Inv. Math. 130 (1997) 1–37.

[17] D. Wu, Kähler–Einstein metrics of negative Ricci curvature on general quasiprojective manifolds, Comm. Anal. Geom. 16 (2008) 395–435.

[18] S.-T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation, I, Comm. Pure Appl. Math. 31 (1978) 339–411.