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Algebra

Chain conditions in special pullbacks

Conditions de chaîne dans des pullbacks particuliers

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ABSTRACT

Let $D \subseteq E$ denote an extension of commutative rings with identity, I be a nonzero proper ideal of D, Γ mean a nonzero torsion-free additive grading monoid with $\Gamma \cap -\Gamma = \{0\}$ and $\Gamma^* = \Gamma \setminus \{0\}$. Let $E[\Gamma]$ be the semigroup ring of Γ over E, $D + E[\Gamma^*] = \{f \in E[\Gamma] \mid f(0) \in D\}$ and $D + I[\Gamma^*] = \{f \in D[\Gamma] \mid$ the coefficients of nonconstant terms of f belong to $I\}$. In this paper, we give some conditions for the rings (resp., domains) $D + E[\Gamma^*]$ and $D + I[\Gamma^*]$ to be Noetherian (resp., to satisfy the ascending chain condition on principal ideals).

RÉSUMÉ

Soient $D \subseteq E$ une extension d'anneaux commutatifs unitaires, I un idéal non nul et propre de D et Γ un monoïde commutatif simplifiable sans torsion non trivial tel que $\Gamma \cap -\Gamma = \{0\}$ et $\Gamma^* = \Gamma \setminus \{0\}$. Soient $E[\Gamma]$ l'anneau semi-groupe de Γ sur E, $D + E[\Gamma^*] = \{f \in E[\Gamma] \mid f(0) \in D\}$ et $D + I[\Gamma^*] = \{f \in D[\Gamma] \mid$ les coefficients des termes non-constants de f appartiennent à $I\}$. Dans cet article, nous donnons certaines conditions pour que les anneaux (resp., domaines) $D + E[\Gamma^*]$ et $D + I[\Gamma^*]$ soient Noethériens (resp., satisfassent la condition de chaîne ascendante sur les idéaux principaux).

Version française abrégée

Soient $D \subseteq E$ une extension d'anneaux commutatifs unitaires, I un idéal non nul et propre de D et Γ un monoïde commutatif simplifiable sans torsion non trivial tel que $\Gamma \cap -\Gamma = \{0\}$ et $\Gamma^* = \Gamma \setminus \{0\}$. Soient $E[\Gamma]$ l'anneau semi-groupe de Γ sur E, $D + E[\Gamma^*] = \{f \in E[\Gamma] \mid f(0) \in D\}$ et $D + I[\Gamma^*] = \{f \in D[\Gamma] \mid$ les coefficients des termes non-constants de f appartiennent à $I\}$. Dans cet article, nous donnons certaines conditions pour que les anneaux (resp., domaines) $D + E[\Gamma^*]$ et $D + I[\Gamma^*]$ soient Noethériens (resp., satisfassent la condition de chaîne ascendante sur les idéaux principaux). Notamment, nous montrons le résultat suivant :

(1) $D + E[\Gamma^*]$ (resp., $D + I[\Gamma^*]$) est un anneau Noethérien si et seulement si D est un anneau Noethérien, E est un D-module de type fini (resp., $I = I^2$) et Γ est de type fini.
(2) Si D et E sont deux anneaux intègres, alors $D + E[\Gamma^*]$ satisfait la condition de la chaîne ascendante sur les idéaux principaux si et seulement si $\bigcap_{n \geqslant 1} a_1 \cdots a_n E = (0)$ pour toute suite (a_n) d'éléments non inversibles de D et Γ satisfait la condition de la chaîne ascendante sur les idéaux principaux. De plus, si D satisfait la condition de la chaîne ascendante

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sur les idéaux principaux et Γ satisfait la condition de la chaîne ascendante sur les idéaux principaux, alors $D + I[\Gamma^*]$ satisfait la condition de la chaîne ascendante sur les idéaux principaux.

1. Introduction

Throughout this paper, $D \subseteq E$ denotes an extension of commutative rings with identity, I is a nonzero proper ideal of D, X is an indeterminate over E, Γ means a nonzero torsion-free (additive) grading monoid with $\Gamma \cap -\Gamma = \{0\}$, and $\Gamma^* = \Gamma \setminus \{0\}$, $E[\Gamma]$ is the semigroup ring of Γ over E, $D + E[\Gamma^*] = \{f \in E[\Gamma] \mid f(0) \in D\}$, and $D + I[\Gamma^*] = \{f \in D[\Gamma] \mid$ the coefficients of nonconstant terms of f belong to $I\}$.

Finiteness conditions have for many years been important tools in commutative algebra and algebraic geometry because of their use in producing many theorems and applications. For example, a relation between the ascending chain conditions on ideals and finitely generatedness of ideals in rings permits an interesting measure of the size and behavior of such rings, and the Noetherian condition plays a significant role to prove many results on varieties, homology and cohomology.

In [8, Proposition 2.1], Hizem showed that $D + XE[X]$ is a Noetherian ring if and only if D is a Noetherian ring and E is a finitely generated D-module, and in [4, Proposition 1.2], Dumitrescu et al. showed that $D + XE[X]$ satisfies the ascending chain condition on principal ideals if and only if $\bigcap_{n\geqslant 1} f_1 \cdots f_n E = (0)$ for each infinite sequence $(f_n)_{n\geqslant 1}$ consisting of (nonzero) nonunits of D. In [3, Propositions 5.6 and 5.7], D'Anna et al. generalized Hizem's result and proved that (1) if A and B are commutative rings with unity, J is an ideal of B and $f : A \to B$ is a ring homomorphism, then the amalgamated algebra $A \bowtie^f J := \{(a, f(a) + j) \mid a \in A$ and $j \in J\}$ is a Noetherian ring if and only if A and $f(A) + J$ are Noetherian rings; and (2) if J is a finitely generated A-module, then $A \bowtie^f J$ is Noetherian if and only if A is Noetherian.

In this article, we study some finite conditions on special kinds of pullbacks which are the so-called $D + E[\Gamma^*]$ and $D + I[\Gamma^*]$ constructions. These composite semigroup rings are also nice examples of (Γ-)graded rings and $D + M$ constructions. In particular, if Γ is the set of nonnegative integers, then $D + E[\Gamma^*] = D + XE[X]$ and $D + I[\Gamma^*] = D + XI[X]$; so $D + E[\Gamma^*]$ guarantees some algebraic properties of intermediate rings between polynomial rings $D[X]$ and $E[X]$, and $D + I[\Gamma^*]$ provides us some information of polynomial rings which are contained in the usual polynomial ring $D[X]$. The main purpose is to show the following:

(1) $D + E[\Gamma^*]$ (resp., $D + I[\Gamma^*]$) is a Noetherian ring if and only if D is a Noetherian ring, E is a finitely generated D-module (resp., $I = I^2$) and Γ is finitely generated.
(2) For integral domains D and E, $D + E[\Gamma^*]$ (resp., $D + I[\Gamma^*]$) satisfies the ascending chain condition on principal ideals if and only if $\bigcap_{n\geqslant 1} a_1 \cdots a_n E = (0)$ for each sequence (a_n) of nonunits of D and Γ satisfies the ascending chain condition on principal ideals. Moreover, if D and Γ satisfy the ascending chain condition on principal ideals, then so does $D + I[\Gamma^*]$.

2. When $D + E[\Gamma^*]$ and $D + I[\Gamma^*]$ are Noetherian rings

In this section, we give necessary and sufficient conditions for the rings $D + E[\Gamma^*]$ and $D + I[\Gamma^*]$ to be Noetherian rings. Before starting, we review the concept of a finitely generated monoid. An additive monoid Γ is said to be *finitely generated* if there exists a finite subset $\{\alpha_1, \dots, \alpha_m\}$ of Γ^* such that $\Gamma = \{\sum_{i=1}^m n_i\alpha_i \mid n_i$ is a nonnegative integer$\}$. In this case, we denote $\Gamma = \langle \alpha_1, \dots, \alpha_m \rangle$.

Our first result is an equivalent condition for the ring $D + E[\Gamma^*]$ to be Noetherian.

Theorem 2.1. *$D + E[\Gamma^*]$ is a Noetherian ring if and only if D is a Noetherian ring, E is a finitely generated D-module and Γ is finitely generated.*

Proof. ($\Rightarrow$) Assume that $D + E[\Gamma^*]$ is a Noetherian ring. Then $D \cong D + E[\Gamma^*]/E[\Gamma^*]$ is a Noetherian ring. Let $\alpha \in \Gamma^*$. Then $X^\alpha E[\Gamma]$ is a finitely generated ideal of $D + E[\Gamma^*]$; so $X^\alpha E[\Gamma] = (g_1, \dots, g_n)$, where each $g_i \in D + E[\Gamma^*]$. Since $g_i \in X^\alpha E[\Gamma]$, $g_i = X^\alpha f_i$ for some $f_i \in E[\Gamma]$. Therefore $X^\alpha E[\Gamma] = X^\alpha f_1(D + E[\Gamma^*]) + \cdots + X^\alpha f_n(D + E[\Gamma^*])$, and hence $E = f_1(0)D + \cdots + f_n(0)D$, which indicates that E is a finitely generated D-module.

Finally, we prove that Γ is finitely generated. Note that since Γ is a torsion-free grading monoid with $\Gamma \cap -\Gamma = \{0\}$, we may assume that $0 \leqslant \alpha$ for all $\alpha \in \Gamma$, where $\leqslant$ is a total order of Γ compatible with its operation [7, Corollary 3.4]. Suppose to the contrary that Γ is not finitely generated and choose $\alpha_1 \in \Gamma^*$. Then $\Gamma \neq \langle \alpha_1 \rangle$. We first claim that $\Gamma^* \neq \alpha_1 + \Gamma$. Suppose that $\Gamma^* = \alpha_1 + \Gamma$. Then α_1 is the smallest element in Γ^*. If there is a $\gamma \in \Gamma^*$ such that $\gamma > n\alpha_1$ for all $n \geqslant 1$, then $\{\gamma - n\alpha_1\}_{n\geqslant 1}$ is an infinite strictly decreasing sequence in Γ^*; so $\{X^{\gamma - n\alpha_1}(D + E[\Gamma^*])\}_{n\geqslant 1}$ is an infinite strictly ascending chain of ideals of $D + E[\Gamma^*]$, which contradicts the fact that $D + E[\Gamma^*]$ is Noetherian. Hence for any $\gamma \in \Gamma^*$, $\gamma \leqslant n\alpha_1$ for some $n \geqslant 1$. Let $\beta \in \Gamma^*$ with $\alpha_1 < \beta \leqslant 2\alpha_1$. Since $\beta = \alpha_1 + \gamma_1$ for some $\gamma_1 \in \Gamma^*$, we have $0 < \gamma_1 \leqslant \alpha_1$. By the minimality of α_1, $\gamma_1 = \alpha_1$; so $\beta = 2\alpha_1$. Assume that there are no members in Γ^* properly between $n\alpha_1$ and $(n+1)\alpha_1$ for $n \geqslant 1$. If $\beta' \in \Gamma^*$ with $(n+1)\alpha_1 < \beta' \leqslant (n+2)\alpha_1$, then $\beta' = \alpha_1 + \gamma'$ for some $\gamma' \in \Gamma^*$; so $n\alpha_1 < \gamma' \leqslant (n+1)\alpha_1$. By the induction hypothesis, $\gamma' = (n+1)\alpha_1$; hence we have $\beta' = (n+2)\alpha_1$. This shows that there are no elements in Γ^* properly between $n\alpha_1$ and $(n+1)\alpha_1$ for all $n \geqslant 1$. Hence $\Gamma = \langle \alpha_1 \rangle$, which is a contradiction. Therefore $\alpha_1 + \Gamma \subsetneq \Gamma^*$, and hence we can take $\alpha_2 \in \Gamma^* \setminus \alpha_1 + \Gamma$. Suppose that we have $\alpha_1, \dots, \alpha_{n+1} \in \Gamma^*$ so that $\alpha_{n+1} \in \Gamma^* \setminus \bigcup_{i=1}^n \alpha_i + \Gamma$, where $n \geqslant 1$. Note that

$\Gamma \neq \langle \alpha_1, \ldots, \alpha_{n+1} \rangle$. Next, we claim that $\Gamma^* \neq \bigcup_{i=1}^{n+1} \alpha_i + \Gamma$. Otherwise, let $\delta \in \Gamma^* \setminus \langle \alpha_1, \ldots, \alpha_{n+1} \rangle$. Then $\delta \in \alpha_i + \Gamma$ for some i; so $\delta = \alpha_i + \gamma_1$ for some $\gamma_1 \in \Gamma$. Since $\delta \notin \langle \alpha_1, \ldots, \alpha_{n+1} \rangle$, $\gamma_1 \in \Gamma^*$. Assume that $\delta = \alpha_{\sigma(1)} + \cdots + \alpha_{\sigma(k)} + \gamma_k$, where $k \geqslant 1$, $\gamma_k \in \Gamma^*$ and $\sigma : \mathbb{N} \to \{1, \ldots, n+1\}$ is a map. Now, $\gamma_k = \alpha_{\sigma(k+1)} + \gamma_{k+1}$ for some $\gamma_{k+1} \in \Gamma$. Since $\delta \notin \langle \alpha_1, \ldots, \alpha_{n+1} \rangle$, $\gamma_{k+1} \in \Gamma^*$. Hence for all $n \geqslant 1$, there exists a $\gamma_n \in \Gamma^*$ such that $\delta = \alpha_{\sigma(1)} + \cdots + \alpha_{\sigma(n)} + \gamma_n$. Note that $(\gamma_n)_{n \geqslant 1}$ is an infinite strictly decreasing sequence in Γ^*; so $\{X^{\gamma_n}(D + E[\Gamma^*])\}_{n \geqslant 1}$ is an infinite ascending chain of ideals of $D + E[\Gamma^*]$. However this is impossible. Therefore $\bigcup_{i=1}^{n+1} \alpha_i + \Gamma \subsetneq \Gamma^*$, and thus we can take $\alpha_{n+2} \in \Gamma^* \setminus \bigcup_{i=1}^{n+1} \alpha_i + \Gamma$. Now, we obtain an infinite strictly ascending chain $\{\langle \alpha_1, \ldots, \alpha_n \rangle\}_{n \geqslant 1}$ of submonoids of Γ. Hence $\{(X^{\alpha_1}, \ldots, X^{\alpha_n})(D + E[\Gamma^*])\}_{n \geqslant 1}$ is an infinite strictly ascending chain of ideals of $D + E[\Gamma^*]$, which is absurd. Thus Γ is finitely generated.

($\Leftarrow$) Consider the pullback diagram $D + E[\Gamma^*]$ given by

$$\begin{tikzcd} D + E[\Gamma^*] \arrow[r] \arrow[d] & D \arrow[d, "u"] \\ E[\Gamma] \arrow[r] & E = E[\Gamma]/E[\Gamma^*]. \end{tikzcd}$$

Since D is a Noetherian ring and E is a finitely generated D-module, u is finite [1, p. 30] and E is a Noetherian D-module. Hence E is a Noetherian ring. Note that $E[\Gamma]$ is a Noetherian ring [7, Theorem 7.7]. Thus $D + E[\Gamma^*]$ is a Noetherian ring [5, Proposition 1.7]. $\square$

Applying Theorem 2.1 to the case when or Γ is a finite product of copies of $\mathbb{N}_0$, we can recover

Corollary 2.2. *(cf. [8, Proposition 2.1]) $D + (X_1, \ldots, X_n)E[X_1, \ldots, X_n]$ is a Noetherian ring if and only if D is a Noetherian ring and E is a finitely generated D-module.*

Recall that a domain R is a *Dedekind domain* if every nonzero ideal of R is invertible. It is well known that R is a Dedekind domain if and only if R is a Noetherian Prüfer domain.

Corollary 2.3. *Let $D \subseteq E$ be an extension of integral domains and K be the quotient field of D. Then the following are equivalent.*

(1) *$D + E[\Gamma^*]$ is a principal ideal domain.*
(2) *$D + E[\Gamma^*]$ is a Dedekind domain.*
(3) *$D = E = K$ and Γ is isomorphic to $\mathbb{N}_0$*, i.e., $D + E[\Gamma^*] = K[X]$.

Proof. (3) $\Rightarrow$ (1) $\Rightarrow$ (2) Clear.

(2) $\Rightarrow$ (3) If $D + E[\Gamma^*]$ is a Dedekind domain, then $D + E[\Gamma^*]$ is a Prüfer domain; so $E = K$ and Γ is a Prüfer submonoid of $\mathbb{Q}$ [9, Theorem 2.4]. Since $D + E[\Gamma^*]$ is Noetherian, by Theorem 2.1, K is a finitely generated D-module; so $K = \frac{a_1}{c}D + \cdots + \frac{a_n}{c}D$ for some $0 \neq c, a_1, \ldots, a_n \in D$. Hence $K = cK = a_1 D + \cdots + a_n D \subseteq D$, and thus $D = K$. Since Γ is a Prüfer submonoid of $\mathbb{Q}$, $\Gamma = \bigcup \langle \beta_i \rangle$ for some ascending chain $\{\langle \beta_i \rangle\}$ of cyclic submonoids of Γ. Since Γ is finitely generated, its generators should be contained in some $\langle \beta_j \rangle$. Thus $\Gamma = \langle \beta_j \rangle$. $\square$

For a nonzero ideal I of D, we give an equivalent condition for the ring $D + I[\Gamma^*]$ to be Noetherian. Gilmer proved that for a commutative ring R and a nonzero ideal I of R, $R + I[\{X_\lambda\}_{\lambda \in \Lambda}]$ is Noetherian if and only if Λ is finite, R is Noetherian, and I is idempotent [6, Theorem 1]. Using the proofs in [6, Theorem 1] and Theorem 2.1, we can easily prove the following and give the proof to make the paper self-contained.

Theorem 2.4. *Let I be a nonzero ideal of D. Then $D + I[\Gamma^*]$ is a Noetherian ring if and only if D is a Noetherian ring, $I^2 = I$ and Γ is finitely generated.*

Proof. ($\Rightarrow$) Suppose that $D + I[\Gamma^*]$ is a Noetherian ring. Then $D \cong (D + I[\Gamma^*])/I[\Gamma^*]$ is a Noetherian ring. Also, a simple modification of the proof of Theorem 2.1 shows that Γ is finitely generated. Let $0 \neq a \in I$ and $\alpha \in \Gamma^*$. Consider the ideal $(aX^\alpha, aX^{2\alpha}, \ldots)$ of $D + I[\Gamma^*]$. Since $D + I[\Gamma^*]$ is Noetherian, $aX^{(m+1)\alpha} \in (aX^\alpha, \ldots, aX^{m\alpha})$ for some positive integer m; so we have $aX^{(m+1)\alpha} = \sum_{i=1}^m s_i a X^{i\alpha}$ for some $s_i \in D + I[\Gamma^*]$. Comparing the coefficients of $X^{(m+1)\alpha}$ from each side of this equality, we conclude $a = ba$ for some $b \in I$. Hence $I \subseteq I^2$, and thus $I = I^2$.

($\Leftarrow$) Suppose that D is a Noetherian ring, $I^2 = I$ and Γ is finitely generated. Then I is principal and is generated by an idempotent element a [6, Lemma 1]. Then $D \cong I \oplus J$, where $J = \{x - ax \mid x \in D\}$ is an ideal of D. Since D is Noetherian, $I \cong D/J$ and $J \cong D/I$ are Noetherian rings, and I is a ring with identity a. Since $\Gamma = \langle \alpha_1, \ldots, \alpha_m \rangle$, $I[\Gamma] = I[X^{\alpha_1}, \ldots, X^{\alpha_m}]$ is an ideal of $D[X^{\alpha_1}, \ldots, X^{\alpha_m}]$ which is Noetherian [7, Theorem 7.7]. Hence $D + I[\Gamma^*] = J \oplus I[\Gamma]$ is a Noetherian ring. $\square$

Remark 2.5. If $D + I[\Gamma^*]$ is an integral domain, then 0 and 1 are the only idempotent elements of $D + I[\Gamma^*]$; so if $D + I[\Gamma^*]$ is a Noetherian domain, then we have either $I = (0)$ or $I = D$ by the proof of Theorem 2.4. Thus, if I is a nonzero proper ideal of an integral domain D, then $D + I[\Gamma^*]$ is never a Noetherian domain.

3. Ascending chain conditions on principal ideals in the domains $D + E[\Gamma^*]$ and $D + I[\Gamma^*]$

In this section, we give equivalent conditions for the domains $D + E[\Gamma^*]$ and $D + I[\Gamma^*]$ to satisfy the ascending chain condition on principal ideals, where $D \subseteq E$ is an extension of integral domains. We say that an integral domain R satisfies the *ascending chain condition on principal ideals* (ACCP) if there does not exist an infinite strictly ascending chain of principal ideals of R. It was shown that R satisfies ACCP if and only if $\bigcap_{n\geqslant 1} a_1 \cdots a_n R = (0)$ for each infinite sequence $(a_n)_{n\geqslant 1}$ of nonunits of R [4, Remark 1.1]. As a semigroup version, we say that a monoid Γ satisfies the *ascending chain condition on principal ideals* (ACCP) if there does not exist an infinite strictly ascending chain of principal ideals of Γ. For an $\alpha \in \Gamma$, $\alpha + \Gamma$ denotes a principal ideal of Γ generated by α.

Lemma 3.1. *Γ satisfies ACCP if and only if $\bigcap_{n\geqslant 1}(\alpha_1 + \cdots + \alpha_n + \Gamma) = \emptyset$, where $\alpha_i \in \Gamma^*$.*

Proof. ($\Rightarrow$) Let $\beta \in \bigcap_{n\geqslant 1}(\alpha_1 + \cdots + \alpha_n + \Gamma)$. Then for each $n \geqslant 1$, $\beta = \alpha_1 + \cdots + \alpha_n + \beta_n$, where $\beta_n \in \Gamma$. We claim that $\beta_1 + \Gamma \subsetneq \beta_2 + \Gamma \subsetneq \beta_3 + \Gamma \subsetneq \cdots$ is an infinite strictly ascending chain of principal ideals of Γ. Since Γ is cancellative, $\beta_n = \alpha_{n+1} + \beta_{n+1}$, so $\beta_1 + \Gamma \subseteq \beta_2 + \Gamma \subseteq \beta_3 + \Gamma \subseteq \cdots$. If $\beta_{n+1} \in \beta_n + \Gamma$, then $\beta_{n+1} = \beta_n + \gamma_n$ for some $\gamma_n \in \Gamma$. Hence $\beta_n = \alpha_{n+1} + \beta_{n+1} = \alpha_{n+1} + \beta_n + \gamma_n$; so $\alpha_{n+1} + \gamma_n = 0$, i.e., α_{n+1} is invertible. This is a contradiction to that $\Gamma \cap -\Gamma = \{0\}$.

($\Leftarrow$) Suppose that there exists an infinite strictly ascending chain $\alpha_1 + \Gamma \subsetneq \alpha_2 + \Gamma \subsetneq \alpha_3 + \Gamma \subsetneq \cdots$ of principal ideals of Γ. Then for each $n \geqslant 1$, there exists a $\gamma_n \in \Gamma^*$ such that $\alpha_n = \alpha_{n+1} + \gamma_n$. Since Γ is cancellative, $\alpha_1 = \gamma_1 + \cdots + \gamma_n + \alpha_{n+1}$ for each $n \geqslant 1$. Hence $\alpha_1 \in \bigcap_{n\geqslant 1}(\gamma_1 + \cdots + \gamma_n + \Gamma)$, which contradicts the assumption. Thus Γ satisfies ACCP. □

Lemma 3.2. *Let $(f_n)_{n\geqslant 1}$ be an infinite sequence of nonunits of $D + E[\Gamma^*]$ containing an infinite subsequence (f_{n_k}) of nonconstant polynomials (i.e., $f_{n_k} \notin D$). If Γ satisfies ACCP, then $\bigcap_{n\geqslant 1} f_1 f_2 \cdots f_n(D + E[\Gamma^*]) = (0)$.*

Proof. Without loss of generality, we may assume that $f_n \notin D$ for all $n \geqslant 1$. For each f_n, let α_n denote the degree of f_n. Suppose that $\bigcap_{n\geqslant 1} f_1 f_2 \cdots f_n(D + E[\Gamma^*]) \neq (0)$, and let $0 \neq g \in \bigcap_{n\geqslant 1} f_1 f_2 \cdots f_n(D + E[\Gamma^*])$ be of degree β. Then $\beta \in \alpha_1 + \cdots + \alpha_n + \Gamma$ for each $n \geqslant 1$, and thus $\beta \in \bigcap_{n\geqslant 1}(\alpha_1 + \cdots + \alpha_n + \Gamma)$. This is a contradiction to Lemma 3.1. □

Lemma 3.2 is not true without the assumption that Γ satisfies ACCP.

Example 3.3. Let $\Gamma = \mathbb{Q}^{\geqslant 0}$ be the set of nonnegative rational numbers. Then Γ is a nonzero torsion-free additive grading monoid which does not satisfy ACCP, because $\frac{1}{p} + \Gamma \subsetneq \frac{1}{p^2} + \Gamma \subsetneq \cdots$, where p is a prime integer. Put $f_n = X^{\frac{1}{p^n}}$ for $n \geqslant 1$. Then $X^{\frac{1}{p-1}} \in \bigcap_{n\geqslant 1} f_1 \cdots f_n(D + E[\Gamma^*])$ for any extension $D \subseteq E$ of domains.

We now give the main result in this section.

Theorem 3.4. *$D + E[\Gamma^*]$ satisfies ACCP if and only if $\bigcap_{n\geqslant 1} a_1 \cdots a_n E = (0)$ for each sequence (a_n) of nonunits of D and Γ satisfies ACCP.*

Proof. If $D + E[\Gamma^*]$ satisfies ACCP, then Γ satisfies ACCP, for if $\alpha_1 + \Gamma \subsetneq \alpha_2 + \Gamma \subsetneq \cdots$ is an infinite strictly ascending chain of principal ideals of Γ, then $(X^{\alpha_1}) \subsetneq (X^{\alpha_2}) \subsetneq \cdots$ is also an infinite strictly ascending chain of principal ideals of $D + E[\Gamma^*]$, which is impossible. Let $(f_n)_{n\geqslant 1}$ be an infinite sequence of nonunits of $D + E[\Gamma^*]$. By Lemma 3.2, without loss of generality, we may assume that $f_n \in D$ for each n. It then follows that

$$\bigcap_{n\geqslant 1} f_1 \cdots f_n\bigl(D + E[\Gamma^*]\bigr) = \bigcap_{n\geqslant 1} f_1 \cdots f_n D + \left(\bigcap_{n\geqslant 1} f_1 \cdots f_n E\right)[\Gamma^*]. \tag{1}$$

Hence $\bigcap_{n\geqslant 1} f_1 \cdots f_n(D + E[\Gamma^*]) = (0)$ if and only if $\bigcap_{n\geqslant 1} f_1 \cdots f_n E = (0)$. □

Remark 3.5. (1) The proof of Theorem 3.4 shows that if $D + E[\Gamma^*]$ satisfies ACCP, then so does D.

(2) In Theorem 3.4, E need not satisfy ACCP. For example, if D is a field, then $D + XE[X]$ always satisfies ACCP [2, Proposition 1.1].

Recall that a domain D is a *generalized unique factorization domain* if every nonzero nonunit of D is expressible as a finite product of mutually coprime prime quanta. In [10, Theorem 1.6(2)], Lim showed that if the quotient group of Γ satisfies the ascending chain condition on cyclic subgroups, then $D + E[\Gamma^*]$ is a generalized unique factorization domain if and only if $D = E$, D is a generalized unique factorization domain and Γ is a weakly factorial GCD-semigroup. Now, we characterize the unique factorization domain of the form $D + E[\Gamma^*]$.

Corollary 3.6. *$D + E[\Gamma^*]$ is a unique factorization domain if and only if $D = E$, D is a unique factorization domain and Γ is a unique factorization semigroup satisfying the ascending chain condition on cyclic submonoids.*

Proof. Before proving this, note that R is a unique factorization domain if and only if R is a GCD-domain satisfying ACCP. (Recall that a domain R is a *GCD-domain* if any two elements in R have a greatest common divisor (equivalently, $aR \cap bR$ is principal for all $a, b \in R$).)

($\Rightarrow$) If $D + E[\Gamma^*]$ is a unique factorization domain, then $D + E[\Gamma^*]$ is a GCD-domain; so $E = D_S$ for some multiplicative subset S of D [9, Theorem 2.5]. If $D \subsetneq E$, then $\frac{1}{s} \notin D$ for some $s \in S$; so $\bigcap_{n\geqslant 1} s^n D_S = (0)$ by Theorem 3.4. However $1 = \frac{s^n}{s^n}$ for each $n \geqslant 1$; so $1 \in \bigcap_{n\geqslant 1} s^n D_S$. This is impossible, and hence $D = E$. Therefore $D[\Gamma]$ is a unique factorization domain, and thus D is a unique factorization domain and Γ is a unique factorization semigroup satisfying the ascending chain condition on cyclic submonoids [7, Theorem 14.7].

($\Leftarrow$) This was shown in [7, Theorem 14.16]. □

A similar argument as in the proof of Theorem 3.4 also shows the following result:

Theorem 3.7. *Let I be a nonzero proper ideal of a domain D. Then $D + I[\Gamma^*]$ satisfies ACCP if and only if D and Γ satisfy ACCP.*

We end this article with an example which shows that the converse of Theorem 3.7 does not hold.

Example 3.8. (1) Let $\mathbb{Z}$ be the ring of integers, $(f_n)_{n\geqslant 1}$ be a sequence of nonzero nonunits in $\mathbb{Z} + (2\mathbb{Z})[\mathbb{Q}_0{}^*]$ and let $0 \neq g \in \bigcap_{n\geqslant 1} f_1 \cdots f_n(\mathbb{Z} + (2\mathbb{Z})[\mathbb{Q}_0{}^*])$. Let a and b_n be the coefficients of initial terms in g and f_n, respectively. Then for each $n \geqslant 1$, $a = b_1 \cdots b_n e_n$ for some $e_n \in \mathbb{Z}$; so almost all b_i should be ± 1. Since f_n is a nonunit, almost all f_i should be nonconstants. Let c and d_n be the coefficients of the highest degree terms in g and f_n, respectively. Then c has infinitely many divisors $d_1, d_2, \ldots$ and almost all d_i are multiples of 2. This is impossible. Thus $\mathbb{Z} + (2\mathbb{Z})[\mathbb{Q}_0{}^*]$ satisfies ACCP.

(2) By Example 3.3, $\mathbb{Q}_0$ does not satisfy ACCP.

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