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Exponential map and L_∞ algebra associated to a Lie pair ☆

Application exponentielle et algèbre L_∞ associée à une paire de Lie

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ABSTRACT

In this Note, we unveil homotopy-rich algebraic structures generated by the Atiyah classes relative to a Lie pair (L, A) of algebroids. In particular, we prove that the quotient L/A of such a pair admits an essentially canonical homotopy module structure over the Lie algebroid A, which we call Kapranov module.

RÉSUMÉ

Dans cette note, nous dévoilons des structures algébriques, riches en homotopies, engendrées par les classes d'Atiyah relatives à une paire de Lie (L, A) d'algébroïdes. En particulier, nous prouvons que le quotient L/A d'une telle paire admet une structure essentiellement canonique de module à homotopie près sur l'algébroïde de Lie A que nous appelons module de Kapranov.

Version française abrégée

Étant donnée une paire de Lie (L, A), c.à.d. une algébroïde de Lie L et une sous-algébroïde de Lie A, la classe d'Atiyah α_E d'un A-module E relative à la paire de Lie (L, A) est définie comme l'obstruction à l'existence d'une L-connexion *A-compatible* sur E. Le lecteur pourrait souhaiter consulter les deux premiers paragraphes de la Section 1 ou la première section de [3] pour un rappel des définitions. Cette classe, dont la définition est fort récente [3], a pour double origine les classes d'Atiyah des fibrés vectoriels holomorphes et les classes de Molino des feuilletages qu'elle généralise. Le quotient L/A d'une paire de Lie est un A-module [3]. Voici une description de sa classe d'Atiyah $\alpha_{L/A}$. La courbure d'une L-connexion ∇ sur L/A (choisie de façon arbitraire) est le morphisme de fibrés $R^\nabla : \wedge^2 L \to \mathrm{End}(L/A)$ défini par la relation $R^\nabla(l_1, l_2) = \nabla_{l_1}\nabla_{l_2} - \nabla_{l_2}\nabla_{l_1} - \nabla_{[l_1,l_2]}$, pour tous $l_1, l_2 \in \Gamma(L)$. Puisque L/A est un A-module, R^∇ s'annule sur $\wedge^2 A$ et, par conséquent, détermine une section $R^\nabla_{L/A}$ du fibré vectoriel $A^* \otimes (L/A)^* \otimes \mathrm{End}(L/A)$. Il fut établi dans [3] que $R^\nabla_{L/A}$ est un 1-cocycle pour l'agébroïde de Lie A à valeurs dans le A-module $(L/A)^* \otimes \mathrm{End}(L/A)$ et que sa classe de cohomologie $\alpha_{L/A} \in H^1(A; (L/A)^* \otimes \mathrm{End}(L/A))$ est indépendante du choix de la connexion.

Définition 0.1. Soit A une algébroïde de Lie sur une variété différentiable M. Un fibré vectoriel $E \to M$ est un module de Kapranov sur A si $\Gamma(\wedge^\bullet A^* \otimes E)$ est une $L_\infty[1]$-algèbre définie par une suite d'applications $\lambda_k : \otimes^k \Gamma(\wedge^\bullet A^* \otimes E) \to \Gamma(\wedge^\bullet A^* \otimes E)[1]$ $(k \in \mathbb{N})$ dont la première $\lambda_1 : \Gamma(\wedge^\bullet A^* \otimes E) \to \Gamma(\wedge^{\bullet+1} A^* \otimes E)$ est la différentielle de Chevalley–Eilenberg

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associée à une action infinitésimale de A sur E et les suivantes sont $\Gamma(\wedge^\bullet A^*)$-multilinéaires. Nous appelerons k-ième crochet de Kapranov l'application λ_k.

Proposition 0.2. *Soit A une algébroïde de Lie sur une variété différentiable M. Un fibré vectoriel E sur M est un module de Kapranov sur A si, et seulement si, A agit infinitésimalement sur E et il existe une suite de morphismes de fibrés vectoriels $R_k : S^k(E) \to A^* \otimes E$ ($k \geqslant 2$) dont la somme $R = \sum_{k=2}^{\infty} R_k \in \Gamma(A^* \otimes \hat{S}(E^*) \otimes E)$ satisfait l'équation de Maurer–Cartan $d_A R + \frac{1}{2}[R, R] = 0$. (Ici, on considère les sections de $\hat{S}(E^*) \otimes E$ comme des champs de vecteurs verticaux formels sur E le long de la section nulle et on en déduit un crochet de Lie naturel sur l'espace vectoriel gradué $\Gamma(A^* \otimes \hat{S}(E^*) \otimes E)$.) Pour tout $k \geqslant 2$, le k-ième crochet de Kapranov λ_k est lié à la k-ième composante $R_k \in \Gamma(A^* \otimes S^k(E^*) \otimes E)$ de l'élément de Maurer–Cartan R au travers de l'identité*

$$\lambda_k(\xi_1 \otimes b_1, \dots, \xi_k \otimes b_k) = (-1)^{|\xi_1|+\dots+|\xi_k|} \xi_1 \wedge \dots \wedge \xi_k \wedge R_k(b_1, \dots, b_k),$$

valide pour tous $b_1, \dots, b_k \in \Gamma(E)$ et tous éléments homogènes $\xi_1, \dots, \xi_k \in \Gamma(\wedge^\bullet A^*)$.

Le théorème qui suit résume notre principal résultat :

Théorème 0.3. *Le quotient L/A d'une paire de Lie (L, A) admet une structure de module de Kapranov sur l'algébroïde de Lie A, canonique à isomorphisme près, dont le $R_2 \in \Gamma(A^* \otimes S^2(L/A)^* \otimes L/A)$ (voir Proposition 1.1) est un cocycle représentant la classe d'Atiyah de L/A relative à la paire (L, A).*

De surcroît, si l'algébroïde L est le fruit de l'accouplement $A \bowtie B$ de l'agébroïde de Lie A avec une autre algébroïde de Lie B telle qu'il existe une B-connexion ∇ sur B sans torsion ni courbure, alors les composantes de l'élément de Maurer–Cartan R sont liées entre elles par la relation de récurrence $R_{k+1} = \partial^\nabla R_k$ où le symbole ∂^∇ désigne la différentielle covariante associée à la connexion ∇.

Comme corollaires, nous retrouvons deux résultats de [3] (*cf.* Corollaires 3.2 et 3.3).

1. Kapranov modules

Let A be a Lie algebroid (either real or complex) over a manifold M with anchor ρ. By an A-module, we mean a module of the corresponding Lie–Rinehart algebra $\Gamma(A)$ over the associative algebra $C^\infty(M)$. An A-connection on a smooth vector bundle E over M is a bilinear map $\nabla : \Gamma(A) \otimes \Gamma(E) \to \Gamma(E)$ satisfying $\nabla_{fa} e = f \nabla_a e$ and $\nabla_a (fe) = (\rho(a) f)e + f \nabla_a e$, for all $a \in \Gamma(A)$, $e \in \Gamma(E)$, and $f \in C^\infty(M)$. A vector bundle E endowed with a *flat* A-connection (also known as an infinitesimal A-action) is an A-module; more precisely, its space of smooth sections $\Gamma(E)$ is one.

Atiyah class Given a Lie pair (L, A) of algebroids, *i.e.* a Lie algebroid L with a Lie subalgebroid A, the Atiyah class α_E of an A-module E relative to the pair (L, A) is defined as the obstruction to the existence of an A-*compatible* L-connection on E. An L-connection ∇ is A-compatible if its restriction to $\Gamma(A) \otimes \Gamma(E)$ is the given infinitesimal A-action on E and $\nabla_a \nabla_l - \nabla_l \nabla_a = \nabla_{[a,l]}$ for all $a \in \Gamma(A)$ and $l \in \Gamma(L)$. This fairly recently defined class (see [3]) has as double origin, which it generalizes, the Atiyah class of holomorphic vector bundles and the Molino class of foliations. The quotient L/A of the Lie pair (L, A) is an A-module [3]. Its Atiyah class $\alpha_{L/A}$ can be described as follows. Choose an L-connection ∇ on L/A extending the A-action. Its curvature is the vector bundle map $R^\nabla : \wedge^2 L \to \mathrm{End}(E)$ defined by $R^\nabla(l_1, l_2) = \nabla_{l_1} \nabla_{l_2} - \nabla_{l_2} \nabla_{l_1} - \nabla_{[l_1, l_2]}$, for all $l_1, l_2 \in \Gamma(L)$. Since L/A is an A-module, R^∇ vanishes on $\wedge^2 A$ and, therefore, determines a section $R^\nabla_{L/A}$ of $A^* \otimes (L/A)^* \otimes \mathrm{End}(L/A)$. It was proved in [3] that $R^\nabla_{L/A}$ is a 1-cocycle for the Lie algebroid A with values in the A-module $(L/A)^* \otimes \mathrm{End}(L/A)$ and that its cohomology class $\alpha_{L/A} \in H^1(A; (L/A)^* \otimes \mathrm{End}(L/A))$ is independent of the choice of the connection.

Kapranov modules over a Lie algebroid Let M be a smooth manifold, and let R be the algebra of smooth functions on M valued in $\mathbb{R}$ (or $\mathbb{C}$). Let A be a Lie algebroid over M. The Chevalley–Eilenberg differential d_A and the exterior product make $\Gamma(\wedge^\bullet A^*)$ into a differential graded commutative R-algebra.

Now let E be a smooth vector bundle over M. Deconcatenation defines an R-coalgebra structure on $\Gamma(S^\bullet E)$. Let $\mathfrak{e}$ denote the ideal of $\Gamma(S^\bullet(E^*))$ generated by $\Gamma(E^*)$. The algebra $\mathrm{Hom}_R(\Gamma(S^\bullet E), R)$ dual to the coalgebra $\Gamma(S^\bullet E)$ is the $\mathfrak{e}$-adic completion of $\Gamma(S^\bullet(E^*))$. It will be denoted by $\Gamma(\hat{S}^\bullet(E^*))$. Equivalently, one can think of the completion $\hat{S}^\bullet(E^*)$ of $S^\bullet(E^*)$ as a bundle of algebras over M. Note that $\Gamma(\wedge^\bullet A^*)$ is an R-subalgebra of $\Gamma(\wedge^\bullet A^* \otimes \hat{S}^\bullet E^*)$.

Recall that an $L_\infty[1]$ algebra is a $\mathbb{Z}$-graded vector space $V = \bigoplus_{n\in\mathbb{Z}} V_n$ endowed with a sequence $(\lambda_k)_{k=1}^\infty$ of skew-symmetric multilinear maps $\lambda_k : \otimes^k V \to V$ of degree 1 satisfying the generalized Jacobi identity

$$\sum_{k=1}^{n} \sum_{\sigma \in \mathfrak{S}_k^{n-k}} \varepsilon(\sigma; v_1, \dots, v_n) \lambda_{1+n-k}\big(\lambda_k(v_{\sigma(1)}, \dots, v_{\sigma(k)}), v_{\sigma(k+1)}, \dots, v_{\sigma(n)}\big) = 0$$

for each $n \in \mathbb{N}$ and for any homogeneous vectors $v_1, v_2, \dots, v_n \in V$. Here $\mathfrak{S}_p^q$ denotes the set of (p,q)-*shuffles*[1] and $\varepsilon(\sigma; v_1, \dots, v_n)$ the *Koszul sign*[2] of the permutation σ of the (homogeneous) vectors $v_1, v_2, \dots, v_n$.

Definition 1.1. A Kapranov module over a Lie algebroid $A \to M$ is a vector bundle $E \to M$ together with an $L_\infty[1]$ algebra structure on $\Gamma(\wedge^\bullet A^* \otimes E)$ defined by a sequence $(\lambda_k)_{k\in\mathbb{N}}$ of multibrackets (called Kapranov multibrackets) such that (1) the unary bracket $\lambda_1 : \Gamma(\wedge^\bullet A^* \otimes E) \to \Gamma(\wedge^{\bullet+1} A^* \otimes E)$ is the Chevalley–Eilenberg differential associated to an infinitesimal A-action on E, and (2) all multibrackets $\lambda_k : \otimes^k \Gamma(\wedge^\bullet A^* \otimes E) \to \Gamma(\wedge^\bullet A^* \otimes E)[1]$ with $k \geqslant 2$ are $\Gamma(\wedge^\bullet A^*)$-multilinear.

Proposition 1.1. *Let A be a Lie algebroid over a smooth manifold M and let E be a smooth vector bundle over M. Each of the following four data is equivalent to a Kapranov A-module structure on E:*

- (i) *A degree 1 derivation D of the graded algebra $\Gamma(\wedge^\bullet A^* \otimes \hat{S}(E^*))$, which preserves the filtration $\Gamma(\wedge A^* \otimes \hat{S}^{\geqslant n}(E^*))$, satisfies $D^2 = 0$, and whose restriction to $\Gamma(\wedge^\bullet A^*)$ is the Chevalley–Eilenberg differential of the Lie algebroid A. (Here, by convention, all elements of $\hat{S}(E^*)$ have degree 0.)*
- (ii) *An infinitesimal action of A on $\hat{S}(E^*)$ by derivations which preserve the decreasing filtration $\hat{S}^{\geqslant n}(E^*)$.*
- (iii) *An infinitesimal action of A on $S(E)$ by coderivations which preserve $S^{\geqslant 1}(E)$ and the increasing filtration $S^{\leqslant n}(E)$.*
- (iv) *An infinitesimal action of A on E together with a sequence of morphisms of vector bundles $\boldsymbol{R}_k : S^k(E) \to A^* \otimes E$ $(k \geqslant 2)$ whose sum $\boldsymbol{R} = \sum_{k=2}^{\infty} \boldsymbol{R}_k \in \Gamma(A^* \otimes \hat{S}(E^*) \otimes E)$ is a solution of the Maurer–Cartan equation $d_A \boldsymbol{R} + \frac{1}{2}[\boldsymbol{R}, \boldsymbol{R}] = 0$. (Here, we consider $\Gamma(\hat{S}(E^*) \otimes E)$ as the space of formal vertical vector fields on E along the zero section and derive a natural Lie bracket on the graded vector space $\Gamma(\wedge^\bullet A^* \otimes \hat{S}(E^*) \otimes E)$.)*

Characterizations (i) and (iv) are related by the identity $D = d_A^{\hat{S}(E^*)} + \boldsymbol{R}$, where $d_A^{\hat{S}(E^*)}$ denotes the Chevalley–Eilenberg differential associated to the infinitesimal A-action on E, and $\boldsymbol{R}$ denotes its own action on $\Gamma(\wedge^\bullet A^* \otimes \hat{S}(E^*))$ by contraction. On the other hand, for any $k \geqslant 2$, the k-th Kapranov multibracket λ_k is related to the k-th component $\boldsymbol{R}_k \in \Gamma(A^* \otimes S^k E^* \otimes E)$ of the Maurer–Cartan element $\boldsymbol{R}$ through the equation

$$\lambda_k(\xi_1 \otimes e_1, \dots, \xi_k \otimes e_k) = (-1)^{|\xi_1|+\dots+|\xi_k|} \xi_1 \wedge \dots \wedge \xi_k \wedge \boldsymbol{R}_k(e_1, \dots, e_k),$$

which is valid for any $e_1, \dots, e_k \in \Gamma(E)$ and any homogeneous elements $\xi_1, \dots, \xi_k$ of $\Gamma(\wedge^\bullet A^*)$.

The algebraic structure described in the above proposition is related to Costello's L_∞ algebras over the differential graded algebra $(\Gamma(\wedge^\bullet A^*), d_A)$ [4], and to Yu's L_∞ algebroids [9].

Two Kapranov A-modules E_1 and E_2 over M are isomorphic if there exists an isomorphism $\Phi : S(E_1) \to S(E_2)$ of bundles of coalgebras over M, which intertwines the infinitesimal A-actions.

2. Exponential map and Poincaré–Birkhoff–Witt isomorphism

Assume $\mathscr{A}$ is a Lie subgroupoid of a Lie groupoid $\mathscr{L}$ (over the same unit space), and let A and L denote the corresponding Lie algebroids. The source map $s : \mathscr{L} \to M$ factors through the quotient of the action of $\mathscr{A}$ on $\mathscr{L}$ by multiplication from the right. Therefore, it induces a surjective submersion $s : \mathscr{L}/\mathscr{A} \to M$. Note that the zero section $0 : M \to L/A$ and the unit section $1 : M \to \mathscr{L}/\mathscr{A}$ are both embeddings of M.

Proposition 2.1. *Each choice of a splitting of the short exact sequence of vector bundles $0 \to A \to L \to L/A \to 0$ and of an L-connection ∇ on L/A extending the A-action determines an exponential map,* i.e. *a fiber bundle map $\exp^\nabla : L/A \to \mathscr{L}/\mathscr{A}$, which identifies the zero section of L/A to the unit section of $\mathscr{L}/\mathscr{A}$, whose differential along the zero section of L/A is the canonical isomorphism between L/A and the tangent bundle to the s-foliation of $\mathscr{L}/\mathscr{A}$ along the unit section, and which is locally diffeomorphic around M.*

Let $\mathcal{N}(L/A)$ denote the space of all functions on L/A which, together with their derivatives of all degrees in the direction of the π-fibers, vanish along the zero section. The space of π-fiberwise differential operators on L/A along the zero section is canonically identified to the symmetric R-algebra $\Gamma(S(L/A))$. Therefore, we have the short exact sequence of R-algebras

$$0 \to \mathcal{N}(L/A) \to C^\infty(L/A) \to \operatorname{Hom}_R\big(\Gamma\big(S(L/A)\big), R\big) \to 0. \tag{1}$$

Likewise, let $\mathcal{N}(\mathscr{L}/\mathscr{A})$ denote the space of all functions on $\mathscr{L}/\mathscr{A}$ which, together with their derivatives of all degrees in the direction of the s-fibers, vanish along the unit section. The space of s-fiberwise differential operators on $\mathscr{L}/\mathscr{A}$ along

[1] A (p,q)-*shuffle* is a permutation σ of the set $\{1, 2, \dots, p+q\}$ such that $\sigma(1) \leqslant \sigma(2) \leqslant \dots \leqslant \sigma(p)$ and $\sigma(p+1) \leqslant \sigma(p+2) \leqslant \dots \leqslant \sigma(p+q)$.

[2] The *Koszul sign* of a permutation σ of the (homogeneous) vectors $v_1, v_2, \dots, v_n$ is determined by the relation $v_{\sigma(1)} \odot v_{\sigma(2)} \odot \dots \odot v_{\sigma(n)} = \varepsilon(\sigma; v_1, \dots, v_n) v_1 \odot v_2 \odot \dots \odot v_n$.

the unit section is canonically identified to the quotient of the enveloping algebra $\mathcal{U}(L)$ by the left ideal generated by $\Gamma(A)$. Therefore, we have the short exact sequence of R-algebras

$$0 \to \mathcal{N}(\mathscr{L}/\mathscr{A}) \to C^\infty(\mathscr{L}/\mathscr{A}) \to \operatorname{Hom}_R\left(\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}, R\right) \to 0. \tag{2}$$

Since the exponential (or more precisely its dual) maps $\mathcal{N}(\mathscr{L}/\mathscr{A})$ to $\mathcal{N}(L/A)$, it induces an isomorphism of R-modules from $\operatorname{Hom}_R(\Gamma(S(L/A)), R)$ to $\operatorname{Hom}_R(\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}, R)$.

Proposition 2.2. *Each choice of a splitting of the short exact sequence of vector bundles $0 \to A \to L \to L/A \to 0$ and of an L-connection ∇ on L/A extending the A-action determines an isomorphism of filtered R-modules $\mathrm{PBW} : \Gamma(S(L/A)) \to \frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$ called Poincaré–Birkhoff–Witt map.*

Remark 2.1. In case $L = A \bowtie B$ is the Lie algebroid sum of a matched pair of Lie algebroids (A, B), the L-connection ∇ on $L/A \cong B$ extending the A-action determines a B-connection on B, the coalgebras $\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$ and $\mathcal{U}(B)$ are isomorphic, and the corresponding Poincaré–Birkhoff–Witt map $\mathrm{PBW} : \Gamma(S(B)) \to \mathcal{U}(B)$ is standard (see [7] for instance).

Proposition 2.3. *The Poincaré–Birkhoff–Witt map associated to a splitting $j : L/A \to L$ of the short exact sequence of vector bundles $0 \to A \to L \to L/A \to 0$ and an L-connection ∇ on L/A satisfies $\mathrm{PBW}(1) = 1$ and, for all $b \in \Gamma(L/A)$ and $n \in \mathbb{N}$, $\mathrm{PBW}(b) = j(b)$ and $\mathrm{PBW}(b^{n+1}) = j(b) \cdot \mathrm{PBW}(b^n) - \mathrm{PBW}(\nabla_{j(b)}(b^n))$, where b^k stands for the symmetric product $b \odot b \odot \cdots \odot b$ of k copies of b.*

Remark 2.2. Although the construction of the Poincaré–Birkhoff–Witt map outlined above presupposes that L and A are integrable real Lie algebroids, PBW can be defined for any real (resp. complex) Lie pair provided one works with local (resp. formal) groupoids.

The infinitesimal actions of A on L/A and $\mathscr{L}/\mathscr{A}$ induce infinitesimal actions of A by derivations on the algebras of functions $C^\infty(L/A)$ and $C^\infty(\mathscr{L}/\mathscr{A})$ and, consequently, on the algebras of infinite jets $\operatorname{Hom}_R(\Gamma(S(L/A)), R)$ and $\operatorname{Hom}_R(\frac{\mathcal{U}(L)}{\mathcal{U}(L)A}, R)$.

Proposition 2.4. (1) *The space $\operatorname{Hom}_R(\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}, R)$ of infinite s-fiberwise jets along M of functions on $\mathscr{L}/\mathscr{A}$ is an associative algebra on which the Lie algebroid A acts infinitesimally by derivations.* (2) *The dual of the exponential map $\mathrm{PBW}^* : \operatorname{Hom}_R(\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}, R) \to \operatorname{Hom}_R(\Gamma(S(L/A)), R)$ is an isomorphism of associative algebras, which may or may not intertwine the infinitesimal A-actions.*

3. $L_\infty[1]$ algebra associated to a Lie pair

Our main result is the following:

Theorem 3.1. *If (L, A) is a Lie pair,* i.e. *a Lie algebroid L together with a Lie subalgebroid A, then L/A admits a Kapranov module structure, canonical up to isomorphism, over the Lie algebroid A, whose $\boldsymbol{R}_2 \in \Gamma(A^* \otimes S^2(L/A)^* \otimes L/A)$ (see Proposition 1.1) is a 1-cocycle representative of the Atiyah class of L/A relative to the pair (L, A).*

Moreover, when $L = A \bowtie B$ is the Lie algebroid sum of a matched pair (A, B) of Lie algebroids and there exists a torsion free flat B-connection ∇ on B, the components of the Maurer–Cartan element $\boldsymbol{R}$ satisfy the recursive formula $\boldsymbol{R}_{k+1} = \partial^\nabla \boldsymbol{R}_k$, where ∂^∇ denotes the covariant differential associated to the connection.

Sketch of proof. Choose a splitting of the short exact sequence of vector bundles $0 \to A \to L \to L/A \to 0$ and an L-connection ∇ on L/A extending the A-action. Identify $\Gamma(\hat{S}(L/A)^*)$ to $\operatorname{Hom}_R(\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}, R)$ via the PBW map and pull back the infinitesimal A-action of the latter to the former. According to Proposition 1.1, the resulting A-action on $\Gamma(\hat{S}(L/A)^*)$ by derivations determines a Kapranov A-module structure on L/A. Making use of Proposition 2.3, one can check directly that $\boldsymbol{R}_2$ is a 1-cocycle representative of the Atiyah class $\alpha_{L/A}$. □

As immediate consequences, we recover the following results of [3]:

Corollary 3.2. *Given a Lie algebroid pair (L, A), let $\mathcal{U}(A)$ denote the universal enveloping algebra of the Lie algebroid A and let $\mathcal{A}$ denote the category of $\mathcal{U}(A)$-modules. The Atiyah class of the quotient L/A makes $L/A[-1]$ into a Lie algebra object in the derived category $D^b(\mathcal{A})$.*

Corollary 3.3. *Let (L, A) be a Lie pair and let $\mathscr{C}$ be a bundle (of finite or infinite rank) of associative commutative algebras on which A acts by derivations. There exists an $L_\infty[1]$ algebra structure on $\Gamma(\wedge^\bullet A^* \otimes L/A \otimes \mathscr{C})$, canonical up to L_∞ isomorphism. Moreover, $H^{\bullet-1}(A; L/A \otimes \mathscr{C})$ is a graded Lie algebra whose Lie bracket only depends on the Atiyah class of L/A.*

For $\mathscr{C} = \mathbb{C}$, the Lie bracket on the cohomology $H^{\bullet-1}(A, L/A)$ happens to be trivial.

4. An example due to Kapranov

Let X be a Kähler manifold with real analytic metric. Recall that the eigenbundles $T_X^{0,1}$ and $T_X^{1,0}$ of the complex structure $J : T_X \to T_X$ ($J^2 = -\mathrm{id}$) form a matched pair of Lie algebroids [6]. Fix a point $x \in X$. The exponential map $\exp_x^{\mathrm{LC}} : T_x X \to X$ defined using the geodesics of the Levi-Civita connection ∇^{LC} originating from the point x needs not be holomorphic.

However, Calabi constructed a holomorphic exponential map $\exp_x^{\mathrm{hol}} : T_x X \to X$ as follows [2] (see also [1]). First, extend the Levi-Civita connection $\mathbb{C}$-linearly to a $T_X \otimes \mathbb{C}$-connection $\nabla^{\mathbb{C}}$ on $T_X \otimes \mathbb{C}$. Since X is Kähler, $\nabla^{\mathrm{LC}} J = 0$ and $\nabla^{\mathbb{C}}$ restricts to a $T_X \otimes \mathbb{C}$-connection on $T_X^{1,0}$. It is easy to check that the induced $T_X^{0,1}$-connection on $T_X^{1,0}$ is the canonical infinitesimal $T_X^{0,1}$-action on $T_X^{1,0}$ — a section of $T_X^{1,0}$ is $T_X^{0,1}$-horizontal iff it is holomorphic — while the induced $T_X^{1,0}$-connection $\nabla^{1,0}$ on $T_X^{1,0}$ is flat and torsion free. Now let X' denote the manifold X and let X'' denote X with the opposite complex structure $-J$. The image of the diagonal embedding $X \hookrightarrow X' \times X''$ is totally real so $X' \times X''$ can be seen as a complexification of X. The restriction of $T_{X' \times X''}$ (resp. its subbundle $T_{X'} \times X''$) along the diagonal X is precisely the complexified tangent bundle $T_X \otimes \mathbb{C}$ (resp. its subbundle $T_X^{1,0}$). (See [8] for a discussion on integration of complex Lie algebroids.) The analytic continuation of the $T_X^{1,0}$-connection $\nabla^{1,0}$ on $T_X^{1,0}$ in a neighborhood of the diagonal is a holomorphic $T_{X'} \times X''$-connection on the Lie algebroid $T_{X'} \times X''$, whose exponential map $\exp_x^{\mathrm{hol}}$ at a diagonal point (x, x) takes $T_x X' \times \{x\}$ (which is $(T_X^{1,0})_x$ or $T_x X$) into $X' \times \{x\}$ (which is X).

Consider the Lie pair $(L = T_{X' \times X''}, A = X' \times T_{X''})$, the corresponding Lie groupoids $\mathscr{L} = (X' \times X'') \times (X' \times X'')$ and $\mathscr{A} = X' \times (X'' \times X'')$, and the associated quotients $L/A = T_{X'} \times X''$ and $\mathscr{L}/\mathscr{A} = (X' \times X') \times X''$. Calabi's holomorphic exponential map $\exp^{\mathrm{hol}}$ is indeed the restriction along the diagonal of the exponential map $\exp^{\nabla^{1,0}} : L/A \to \mathscr{L}/\mathscr{A}$ associated to the $T_{X'} \times X''$-connection $\nabla^{1,0}$ on the Lie algebroid $T_{X'} \times X''$ as described in Proposition 2.1.

Taking the infinite jet of $\exp^{\mathrm{hol}}$, we obtain, as in Proposition 2.2, a Poincaré–Birkhoff–Witt map $\mathrm{PBW}^{\mathrm{hol}} : \Gamma(S(T_X^{1,0})) \to \mathcal{U}(T_X^{1,0})$. Then, pulling back the infinitesimal $T_X^{0,1}$-action on $\mathcal{U}(T_X^{1,0})$ to an infinitesimal $T_X^{0,1}$-action by coderivations on $\Gamma(S(T_X^{1,0}))$, we obtain, as in Theorem 3.1, a Kapranov $T_X^{0,1}$-module structure on $T_X^{1,0}$. In this context, the tensors $R_n \in \Omega^{0,1}(\mathrm{Hom}(S^n T_X^{1,0}, T_X^{1,0}))$ are the curvature $R_2 \in \Omega^{1,1}(\mathrm{End}(T_X^{1,0}))$ and its higher covariant derivatives. Hence we recover the following result of Kapranov:

Theorem 4.1. *([5]) The Dolbeault complex $\Omega^{0,\bullet}(T_X^{1,0})$ of a Kähler manifold is an $L_\infty[1]$ algebra. For $n \geqslant 2$, the n-th multibracket $\lambda_n : \Omega^{0,j_1}(T_X^{1,0}) \otimes \cdots \otimes \Omega^{0,j_n}(T_X^{1,0}) \to \Omega^{0,j_1+\cdots+j_n+1}(T_X^{1,0})$ is the composition of the wedge product with the map associated to $R_n \in \Omega^{0,1}(\mathrm{Hom}(\otimes^n T_X^{1,0}, T_X^{1,0}))$ in the obvious way, while λ_1 is the Dolbeault operator $\bar{\partial} : \Omega^{0,j}(T_X^{1,0}) \to \Omega^{0,j+1}(T_X^{1,0})$.*

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