Partial Differential Equations/Numerical Analysis

The vanishing viscosity as a selection principle for the Euler equations: The case of 3D shear flow

La viscosité évanescente comme critère de sélection pour les solutions de l'équation d'Euler : Le cas du flot de cisaillement

Claude Bardos [a], Edriss S. Titi [b,c], Emil Wiedemann [d]

[a] *Laboratoire Jacques-Louis-Lions, 4, place Jussieu, 75005 Paris, France*
[b] *Department of Mathematics, University of California, Irvine, CA 92697, USA*
[c] *Faculty of Mathematics and Computer Science, The Weizmann Institute of Science, Rehovot 76100, Israel*
[d] *Department of Mathematics, University of British Columbia, and Pacific Institute for the Mathematical Sciences, Vancouver, BC, Canada*

A B S T R A C T

We show that for a certain family of initial data, there exist non-unique weak solutions to the 3D incompressible Euler equations satisfying the weak energy inequality, whereas the weak limit of every sequence of Leray–Hopf weak solutions for the Navier–Stokes equations, with the same initial data, and as the viscosity tends to zero, is uniquely determined and equals the shear flow solution of the Euler equations corresponding to this initial data. This simple example suggests that, also in more general situations, the vanishing viscosity limit of the Navier–Stokes equations could serve as a uniqueness criterion for weak solutions of the Euler equations.

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R É S U M É

On montre que pour une certaine famille de données initiales, il existe plusieurs solutions faibles de l'équation d'Euler incompressible qui satisfont l'inégalité d'énergie au sens faible. Cependant toute solution faible de l'équation d'Euler qui de surcroit est limite faible d'une suite de solutions des équations de Navier–Stokes (au sens de Leray–Hopf) avec les mêmes données initiales et une viscosité évanescente est déterminée de manière unique. Cet exemple simple suggère que, de même, dans des situations plus générales, la limite pour viscosité évanescente des solutions d'équations de Navier–Stokes puisse servir de critère d'unicité pour les solutions faibles des équations d'Euler.

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1. Introduction

Consider the incompressible Euler equations,

$$\partial_t v + v \cdot \nabla v + \nabla p = 0$$
$$\operatorname{div} v = 0, \tag{1}$$

on the d-dimensional torus $\mathbb{T}^d = (-1/2, 1/2)^d$, $d \geqslant 2$. A weak solution (i.e. a solution in the sense of distributions) $v \in C([0, T]; L^2_w(\mathbb{T}^d))$ is said to be *admissible* if it satisfies the weak energy inequality, i.e. if

$$\frac{1}{2} \int_{\mathbb{T}^d} \left| v(x, t) \right|^2 \mathrm{d}x \leqslant \frac{1}{2} \int_{\mathbb{T}^d} \left| v(x, 0) \right|^2 \mathrm{d}x \tag{2}$$

for every $t \in [0, T]$. Here, $T \in (0, \infty]$ and $C([0, T]; L^2_w(\mathbb{T}^d))$ denotes the space of vector fields that are continuous as maps from $[0, T]$ into $L^2(\mathbb{T}^d)$ with respect to the weak topology in L^2.

It has been established in [4] and [11] (see Corollary 3 therein) that there exists a large set of initial data which admit infinitely many admissible weak solutions of (1). Notice that the cited results can be adapted to the case of periodic boundary conditions, considered here, in a straightforward manner. These initial data, however, are constructed in a rather abstract way. Recently, L. Székelyhidi Jr. [10] exhibited the first example of concretely given initial data with the above mentioned non-uniqueness property. More precisely, consider the flat vortex sheet in two dimensions given by

$$V_0(x) = \begin{cases} \vec{e}_1 & \text{if } x_2 > 0 \\ -\vec{e}_1 & \text{if } x_2 < 0. \end{cases} \tag{3}$$

Obviously, the stationary solution $v(\cdot, t) = V_0(\cdot)$ for all $t > 0$ satisfies the Cauchy problem for the Euler equations in the weak sense. Székelyhidi's result can be stated as follows:

Theorem 1. *There exist $T > 0$ and infinitely many weak solutions to the Euler equations in $\mathbb{T}^2 \times [0, T]$ with initial data V_0, given in (3), and pressure zero. Among these, infinitely many conserve the kinetic energy in time, and infinitely many have strictly decreasing energy.*

Here, the kinetic energy is defined as in Eq. (2) above. Székelyhidi also observed that, clearly, any sequence of the Leray–Hopf weak solutions of the Navier–Stokes equations, corresponding to the initial data (3), converges weak-∗ (in fact even strongly) in $L^\infty([0, T]; L^2(\mathbb{T}^2))$ to the stationary solution of the Euler equations, as the viscosity tends to zero. Hence, being a vanishing viscosity limit distinguishes the stationary solution from all the other weak solutions of the Euler equations stated in Theorem 1. The aim of the present note is to prove a similar statement for the slightly more sophisticated case of the three-dimensional shear flow.

2. 3D shear flow

In three dimensions, any initial data of the form $v_0(x) = (v_1(x_2), 0, v_3(x_1, x_2))$ has the shear flow solution of the Euler equations given by

$$v(x, t) = \left(v_1(x_2), 0, v_3\left(x_1 - t v_1(x_2), x_2\right) \right) \tag{4}$$

(see [2] and [6] for results concerning this shear flow). Note in particular that $v(x, t)$ is periodic in x if v_0 is.

Székelyhidi's result can now easily be extended to the case of three-dimensional shear flows as follows:

Corollary 2. *Let $v_0(x) = (v_1(x_2), 0, v_3(x_1, x_2))$, where*

$$v_1(x_2) = \begin{cases} 1 & \text{if } 0 < x_2 < 1/2 \\ -1 & \text{if } -1/2 < x_2 < 0, \end{cases}$$

extended periodically, with basic period $(-1/2, 1/2)$, and $v_3 \in L^2_{x_1, x_2}(\mathbb{T}^2)$ is arbitrary. Then there exist $T > 0$, and infinitely many admissible weak solutions of the 3D Euler equations on $\mathbb{T}^3 \times [0, T]$ with initial data v_0.

Proof. Take $u(x, t) = (u_1(x_1, x_2, t), u_2(x_1, x_2, t))$ to be a solution to the 2D vortex sheet problem as in Theorem 1. Then, the triple

$$\left(u_1(x_1, x_2, t), u_2(x_1, x_2, t), w(x_1, x_2, t) \right)$$

will be a weak solution of the 3D Euler equations (with zero pressure and initial data v_0) if w is a weak solution of the 2D transport equation

$$\partial_t w + u \cdot \nabla w = 0$$
$$w(t = 0) = v_3. \tag{5}$$

Such a solution $w \in L^\infty((0, T); L^2(\mathbb{T}^2))$ exists; see Proposition II.1 in [5], which clearly holds also in the periodic setting. Moreover, we may even assume $w \in C((0, T); L^2_w(\mathbb{T}^2))$, see Appendix A of [4]. Finally, as one can see from the proof of the cited Proposition II.1, we have

$$\big\|w(\cdot,t)\big\|_{L^2(\mathbb{T}^2)} \leqslant \|v_3\|_{L^2(\mathbb{T}^2)} \quad \text{for every } t > 0$$

(this is due to the weak lower semi-continuity of the norm in $L^\infty((0,T); L^2(\mathbb{T}^2))$). Hence our solution (u_1, u_2, w) is an admissible weak solution of the 3D Euler equations. $\square$

Remark 3. It is not possible to deduce from (5) that $\|w(\cdot,t)\|_{L^2}$ is conserved in time: For this to hold, w would have to be a renormalised solution in the sense of DiPerna and Lions [5], which is ruled out by the irregularity of Székelyhidi's solutions.

Before we state and prove the main result of this note, we need an auxiliary result:

Lemma 4. *Let $v \in L^2(\mathbb{T}; \mathbb{R})$ and $w_0 \in L^2(\mathbb{T}^2)$. Then the Cauchy problem for the linear transport equation*

$$\partial_t w(x_1, x_2, t) + v(x_2)\partial_{x_1} w(x_1, x_2, t) = 0$$

$$w(\cdot, 0) = w_0$$

has a solution $w \in C([0,T]; L^2_w(\mathbb{T}^2))$, satisfying the equation in the sense of distributions, and this solution is unique in the class $L^\infty((0,T); L^2(\mathbb{T}^2))$.

We omit the elementary proof of the lemma.

Theorem 5. *Let again $v_0(x) = (v_1(x_2), 0, v_3(x_1, x_2))$, where we assume $v_1 \in L^2(\mathbb{T})$ and $v_3 \in L^2(\mathbb{T}^2)$. Then, for every viscosity $\nu > 0$, there exists a unique Leray–Hopf weak solution of the Navier–Stokes equations with viscosity ν and initial data v_0, and these solutions u^ν converge weak-$*$ in $L^\infty([0,T]; L^2(\mathbb{T}^3))$ to the shear flow (4) corresponding to v_0, as $\nu \to 0$.*

Proof. Let $\nu > 0$. The intuition that the solution of Navier–Stokes should preserve the particular structure of the initial data leads us to the ansatz

$$u^\nu(x, t) = \big(u_1^\nu(x_2, t), 0, u_3^\nu(x_1, x_2, t)\big)$$

and $p^\nu = 0$, where p^ν denotes the pressure. Inserting this into the Navier–Stokes equations gives the so called two-and-half Navier–Stokes equations

$$\partial_t u_1^\nu(x_2, t) - \nu \partial_{x_2}^2 u_1^\nu(x_2, t) = 0,$$

$$\partial_t u_3^\nu(x_1, x_2, t) + u_1^\nu(x_2, t)\partial_{x_1} u_3^\nu(x_1, x_2, t) - \nu \Delta_{x_1, x_2} u_3^\nu(x_1, x_2, t) = 0,$$

which is known to be globally well-posed for this kind of initial data (see, e.g., [6], and see [3] for the global existence and uniqueness of weak solutions of the 2D Navier–Stokes equations). For completion, we observe that the first equation is simply the one-dimensional heat equation with initial data $v_1(x_2)$, whose solution obviously converges to the time-independent function $v_1(x_2)$ strongly in $L^2(\mathbb{T} \times [0, T])$, as the viscosity tends to zero. The second equation is an advection-diffusion equation. By standard parabolic theory (see e.g. Theorem 5 in Section 7.1.3 of [7], which again can be adapted to the periodic case, or the 2D Navier–Stokes theory) there exists a (unique) solution

$$u_3^\nu \in L^2\big([0,T]; H^1(\mathbb{T}^2)\big) \cap C\big([0,T]; L^2(\mathbb{T}^2)\big)$$

(recall that the initial data v_3 is in L^2). Hence, for every fixed $\nu > 0$, we obtain a Leray–Hopf weak solution with the initial data $v_0(x)$. Following ideas from [9] (see, e.g., [1] and [8] for details) one can show that this solution is unique within the class of all 3D Leray–Hopf weak solutions; moreover, this solution depends continuously on the initial data, when the initial data is perturbed in the $L^2(\mathbb{T}^3)$ norm (see [1]). Furthermore, since the family of unique solutions, u^ν, is uniformly bounded in $L_t^\infty L_x^2$, there exists a subsequence u^{ν_k} which converges weak-$*$ to $u \in L_t^\infty L_x^2$, and u satisfies

$$u_1(x_2, t) = v_1(x_2)$$

$$u_2 = 0$$

$$\partial_t u_3(x_1, x_2, t) + v_1(x_2)\partial_{x_1} u_3(x_1, x_2, t) = 0$$

$$u_3(x_1, x_2, 0) = v_3(x_1, x_2). \tag{6}$$

Indeed, the equation for u_3 follows from $u_1^\nu u_3^\nu \overset{*}{\rightharpoonup} u_1 u_3$, thanks to the strong convergence of u_1^ν to u_1. Next, it follows from Lemma 4 above that system (6) has a *unique* solution, and that this unique solution is given precisely by the shear flow (4) (see [2]). Finally, this uniqueness implies that the whole sequence u^ν, and not just a subsequence, converges to the shear flow solution (4).

Combining the corollary and the proposition, we see that among the infinitely many admissible solutions of the Euler equations that are corresponding to the shear flow initial data, $v_0(x) = (v_1(x_2), 0, v_3(x_1))$, the shear flow solution given by (4) has the exclusive property of being a vanishing viscosity limit. $\square$

References

[1] Claude Bardos, Milton Lopes Filho, Dongjuan Niu, Helena Nussenzveig Lopes, Edriss S. Titi, Stability of viscous, and instability of non-viscous, 2D weak solutions of incompressible fluids under 3D perturbations, Preprint, http://arxiv.org/pdf/1201.2742v2.

[2] Claude Bardos, Edriss S. Titi, Loss of smoothness and energy conserving rough weak solutions for the 3D Euler equations, Discrete Contin. Dyn. Syst. Ser. 3 (2) (2010) 185–197.

[3] Peter Constantin, Ciprian Foias, Navier–Stokes Equations, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1988.

[4] Camillo de Lellis, László Székelyhidi Jr., On admissibility criteria for weak solutions of the Euler equations, Arch. Ration. Mech. Anal. 195 (1) (2010) 225–260.

[5] R.J. DiPerna, P.-L. Lions, Ordinary differential equations, transport theory and Sobolev spaces, Invent. Math. 98 (3) (1989) 511–547.

[6] Ronald J. DiPerna, Andrew J. Majda, Oscillations and concentrations in weak solutions of the incompressible fluid equations, Comm. Math. Phys. 108 (4) (1987) 667–689.

[7] Lawrence C. Evans, Partial Differential Equations, second edition, Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 2010.

[8] Dragoş Iftimie, Geneviève Raugel, Some results on the Navier–Stokes equations in thin 3D domains, in: Special issue in celebration of Jack K. Hale's 70th birthday, Part 4, Atlanta, GA/Lisbon, 1998, J. Differential Equations 169 (2) (2001) 281–331.

[9] James Serrin, The initial value problem for the Navier–Stokes equations, in: Nonlinear Problems, Proc. Sympos., Madison, Wis., 1962, Univ. of Wisconsin Press, Madison, Wis., 1963, pp. 69–98.

[10] László Székelyhidi, Weak solutions to the incompressible Euler equations with vortex sheet initial data, C. R. Acad. Sci. Paris, Ser. I 349 (19–20) (2011) 1063–1066.

[11] László Székelyhidi Jr., Emil Wiedemann, Young measures generated by ideal incompressible fluid flows, Arch. Ration. Mech. Anal. 206 (2012) 333–366.