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Some unilateral Korn inequalities with application to a contact problem with inclusions

Quelques inégalités de Korn unilatérales et leur application à un problème de contact avec inclusions

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ABSTRACT

In the first part of this note, some unilateral inequalities of the Korn type are established. These inequalities seem to be new.
In the second part, these inequalities are used in an essential way to prove the existence of a solution (which is not necessarily unique) for a unilateral contact problem involving a matrix material with inclusions of various shapes (the conditions depend on the shape of each inclusion in a remarkable way).

RÉSUMÉ

Dans une première partie sont démontrées quelques inégalités de Korn unilatérales qui semblent nouvelles.
Ces inégalités sont alors utilisées de façon essentielle dans la démonstration de l'existence d'une solution pour un problème de contact unilatéral dans une matrice avec des inclusions de diverses formes (les conditions dépendent de façon remarquable de la forme de chaque inclusion).

Version française abrégée

Dans la première partie de cette note, on démontre quelques inégalités de Korn unilatérales qui semblent nouvelles (Propositions 2.8 et 2.9), ne faisant intervenir sur le bord de l'ouvert considéré que la partie positive de la composante normale de la déformation (et, uniquement pour des inclusions de forme sphérique ou cylindrique, la composante tangentielle de la déformation).

Ces inégalités sont utilisées dans la seconde partie pour traiter un problème de contact unilatéral dans une matrice avec des inclusions de diverses formes. Il s'agit alors d'une inequation variationnelle de type Signorini sans frottement ou avec frottement de type Tresca (Problem $\mathcal{P}'$). Celle-ci s'exprime de manière équivalente sous forme de la minimisation d'une fonctionnelle convexe sur un convexe fermé (Problem $\mathcal{P}$).

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Le résultat d'existence est donné dans la Proposition 3.2. Les hypothèses spécifiques portent sur le second membre uniquement dans les inclusions de forme sphérique ou cylindrique (hypothèses $(\mathcal{H}^j)$). L'unicité n'est pas nécessairement garantie dans les inclusions à cause de l'absence de stricte convexité de la fonctionnelle qui est minimisée.

1. Introduction

Contact problems in the framework of three-dimensional linear elasticity have been studied by many authors. The first treatment of the subject appeared in the papers of Fichera [3–5]. We also refer to the book of G. Duvaut and J.-L. Lions [2] and references therein. More recently J. Nečas and his co-workers have studied the numerical approximation of such problems [7]. See the book of I. Hlaváček, J. Haslinger, J. Nečas and J. Lovíšek [8] for references on the subject.

In this note, we consider a Signorini problem with possible Tresca friction for inclusions surrounded by a matrix. The problem here differs from the previous references insofar as the contact conditions are posed on the full boundaries of each inclusion. The conditions for the existence of the solution(s) are related to those of [3–5], but explicitly involve the moments of the right-hand side with respect to spherical and cylindrical inclusions only (hypotheses $(\mathcal{H}^j)$).

The result is based on a series of unilateral Korn inequalities which seem to be new and which are adapted to the various shapes of inclusions.

1.1. Notations

- The normal component of a vector field v on the boundary of a domain is denoted v_ν, while the tangential component $v - v_\nu \nu$ is denoted v_τ (here ν is the outward unit normal to the boundary of the domain);
- the strain tensor of a vector field v is denoted by $e(v)$;
- the kernel of the strain operator e in a connected domain is the six-dimensional space of rigid motions denoted $\mathcal{R}$:

$$\mathcal{R} \doteq \{x \mapsto v_{a,b}(x) = a \wedge x + b;\ a \text{ and } b \in \mathbb{R}^3\}. \tag{1}$$

2. Korn inequalities

2.1. Classical Korn inequalities

Definition 2.1. A bounded domain O is a *Korn domain* whenever it satisfies the second Korn inequality, i.e., if there exists a constant C such that

$$\forall v \in H^1(O), \quad |v|_{H^1(O)} \leqslant C\big(|v|_{L^2(O)} + \big|e(v)\big|_{L^2(O)}\big). \tag{2}$$

It is known (see Gobert [6]) that this holds for every connected bounded domain with Lipschitz boundary, but is also true for more general domains since a finite union of Korn domains is a Korn domain.

Definition 2.2. A bounded and connected domain O is a *Korn–Wirtinger* domain whenever it satisfies the following Korn-type inequality (similar to the Poincaré–Wirtinger inequality for scalar fields):

there exists a constant C such that for every $v \in H^1(O)$ there is an element $r_v \in \mathcal{R}$ with

$$|v - r_v|_{H^1(O)} \leqslant C\big|e(v)\big|_{L^2(O)}. \tag{3}$$

Examples of Korn–Wirtinger domains are given by the following proposition which is proved by a standard contradiction argument.

Proposition 2.3. *Let O be a* connected *Korn domain for which the embedding from $H^1(O)$ into $L^2(O)$ is* compact. *Then O is a* Korn–Wirtinger *domain.*

The following is a straightforward application of inequality (3):

Corollary 2.4. *Let O be a Korn–Wirtinger domain and Γ be a closed subset of ∂O with non-zero boundary measure. Then the first Korn inequality is satisfied for vector fields which vanish on Γ, i.e. there exists a constant C such that*

$$\forall v \in H^1(O; \Gamma), \quad |v|_{H^1(O)} \leqslant C\big|e(v)\big|_{L^2(O)}. \tag{4}$$

2.2. Two types of nonlinear Korn inequalities

We distinguish two classes of *connected* bounded domains according to a geometric property (of their shapes).

Definition 2.5. A locked domain is a bounded connected domain with Lipschitz boundary for which the only rigid motion which is tangent on its boundary is zero.

Remark 2.6. The only non-locked bounded Lipschitz domains (in $\mathbb{R}^3$) are the spherical balls and the bounded right-sections of circular cylindric bars, for which the tangent rigid motions are obvious.

We start with a simple remark concerning the rigid motions in a bounded domain with a Lipschitz boundary (since a rigid motion is divergence-free).

Lemma 2.7. *Let O be a bounded connected and Lipschitz domain. Then,*

$$r \mapsto \|r\| \doteq |r_\tau|_{L^1(\partial O)} + \left|(r_\nu)^+\right|_{L^1(\partial O)} \quad \text{is a norm on } \mathcal{R}. \tag{5}$$

If O is a locked domain then

$$r \mapsto \|r\|_l \doteq \left|(r_\nu)^+\right|_{L^1(\partial O)} \quad \text{is a norm on } \mathcal{R}. \tag{6}$$

Furthermore, if O is a spherical ball centered at the origin, there exists a constant C such that for all r in $\mathcal{R}$, there exists a vector $b(r)$ in $\mathbb{R}^3$ with

$$\left|r - b(r) \wedge Id\right|_{\mathcal{R}} \leqslant C\left|(r_\nu)^+\right|_{L^1(\partial O)}. \tag{7}$$

Similarly, if O is a right-section of a circular cylinder with axis going through the origin and with unit vector d_O, there exists a constant C such that for all r in $\mathcal{R}$, there is a scalar $\ell(r)$ in $\mathbb{R}$ with

$$\left|r - \ell(r)\, d_O \wedge Id\right|_{\mathcal{R}} \leqslant C\left|(r_\nu)^+\right|_{L^1(\partial O)}. \tag{8}$$

The previous lemma yields two new types of Korn inequalities. It is of interest that they only involve the *positive part* v_ν^+ of the normal component of the trace of v on the boundary so that they are, in some sense, unilateral. Obviously, applying them to $-v$ would involve v_ν^- instead.

Proposition 2.8 *(Unilateral Korn inequality for locked domains). Let O be a locked domain. Then there exists a constant C_1 such that*

$$\forall v \in H^1(O), \quad |v|_{H^1(O)} \leqslant C_1\left(\left|e(v)\right|_{L^2(O)} + \left|(v_\nu)^+\right|_{L^1(\partial O)}\right). \tag{9}$$

Proposition 2.9 *(Unilateral Korn inequalities for non-locked domains). Let O be a connected non-locked domain with Lipschitz boundary. Then there exists a constant C_2 such that*

$$\forall v \in H^1(O), \quad |v|_{H^1(O)} \leqslant C_2\left(\left|e(v)\right|_{L^2(O)} + \left|(v_\nu)^+\right|_{L^1(\partial O)} + \left|(v_\tau)\right|_{L^1(\partial O)}\right). \tag{10}$$

Furthermore, if O is a ball centered at the origin, there exists a constant C_3 and a map $b : H^1(O) \to \mathbb{R}^3$ such that for every v in $H^1(O)$

$$\left|v - b(v) \wedge Id\right|_{H^1(O)} \leqslant C_3\left(\left|e(v)\right|_{L^2(O)} + \left|(v_\nu)^+\right|_{L^1(\partial O)}\right). \tag{11}$$

If O is a right-section of a circular right-cylinder with axis going through the origin and unit vector b, there is a constant C_3 and map $\ell : H^1(O) \to \mathbb{R}$ such that

$$\left|v - \ell(v) b \wedge Id\right|_{H^1(O)} \leqslant C_3\left(\left|e(v)\right|_{L^2(O)} + \left|(v_\nu)^+\right|_{L^1(\partial O)}\right). \tag{12}$$

3. A linear elasticity contact problem in a domain with inclusions

3.1. The problem $\mathcal{P}$

Let Ω be a bounded domain in $\mathbb{R}^3$ with Lipschitz boundary, and Γ_D be a non-empty open subset of the boundary $\partial\Omega$ (Γ_D is the set where a homogeneous Dirichlet condition will be prescribed).

Inside Ω are given a finite number m of open subsets $\Omega^1, \dots, \Omega^m$ which are relatively compact in Ω. Their respective boundaries denoted S^j are assumed to be Lipschitz. Let Ω^0 be the subset $\Omega \setminus \bigcup_{j=1,m} \overline{\Omega}^j$ (see Fig. 1). Finally, we denote

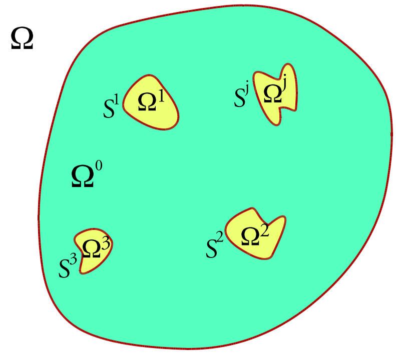

Fig. 1. The various subdomains of Ω.

by Ω^* the open set $\Omega \setminus \bigcup_{j=1,\dots,m} S^j = \bigcup_{j=0,\dots,m} \Omega^j$. The unit normal vector to S^j is denoted by ν and is normalized as inwards for Ω^0.

Consider a symmetric bilinear form:

$$\mathbf{a}\big(e(u), e(v)\big) \doteq \sum_{j=0}^{m} \int_{\Omega^j} \sum_{\alpha,\beta,\gamma,\delta=1}^{3} a_{\alpha\beta\gamma\delta}(x) e(u)_{\gamma\delta}(x) e(v)_{\alpha\beta}(x)\,\mathrm{d}x,$$

where the tensor field $a = (a_{\alpha\beta\gamma\delta})$ has the usual properties of symmetry, boundedness and coercivity (with constant $\overline{\alpha} > 0$) when operating on symmetric tensors of order two, i.e. (with the summation convention):

$$a_{\alpha\beta\gamma\delta} = a_{\beta\alpha\gamma\delta} = a_{\alpha\beta\delta\gamma} = a_{\gamma\delta\alpha\beta},$$
$$\max_{\alpha\beta\gamma\delta} \big|a^{\varepsilon}_{\alpha\beta\gamma\delta}\big|_{L^\infty(\Omega)} < \infty, \qquad \overline{\alpha}\eta_{\alpha\beta}\eta_{\alpha\beta} \leqslant a^{\varepsilon}_{\alpha\beta\gamma\delta}\eta_{\alpha\beta}\eta_{\gamma\delta}.$$

Let $\mathcal{K}$ be the convex set defined, for given non-negative g^j on S^j, by

$$v \in \mathcal{K} \doteq \big\{ v = \big(v^0, \dots, v^m\big) \bigm| v \in H^1\big(\Omega^0\big) \times \dots \times H^1\big(\Omega^m\big),$$
$$v^0{}_{|\Gamma_D} \equiv 0,\ v^j_\nu - v^0_\nu \leqslant g^j \text{ on } S^j \big\}.$$

The vector fields v are the admissible deformation fields with respect to the reference configuration Ω. By standard trace theorems, the jump $[v]_{S^j}$ of v across each surface S^j exists in $H^{1/2}(S^j)$ (it is $v^j_{|S^j} - v^0_{|S^j}$).

The tensor field $\boldsymbol{\sigma}_{\alpha\beta}(v) \doteq \sum_{\gamma,\delta=1}^{3} a_{\alpha\beta\gamma\delta} e(v)_{\gamma\delta}$ is the stress tensor associated to the deformation v and is defined in Ω^* (not to be confused with the surface measures $\mathrm{d}\sigma$!).

With the choice of the unit normals ν, the functions g^j's are the original gaps (in the reference configuration), and the corresponding inequalities represent the non-penetration conditions. In case there is contact in the reference configuration, these functions are just 0. More generally, we will assume that $g^j \in L^1(S^j)$ for $j = 0, \dots, m$ (which implies that the original gaps have finite volumes).

Clearly, $\mathcal{K}$ is a closed convex subset of the space $H^1(\Omega^0) \times \dots \times H^1(\Omega^m)$.

Consider also a family of convex functions Ψ^j, $1 \leqslant j \leqslant m$, where Ψ^j is continuous on $H^{1/2}(S^j)$ and is such that there exist a non-negative number M^j with

$$\Psi^j(w) \geqslant M^j |w_\tau|_{L^1(S^j)} \quad \text{for } |w_\tau|_{L^1(S^j)} \text{ large.} \tag{13}$$

In the case of Tresca friction, the functions Ψ^j are actually explicitly given by

$$\Psi^j(v) \doteq \int_{S^j} G^j(x) \big|v^j_\tau(x) - v^0_\tau(x)\big|\,\mathrm{d}\sigma(x), \tag{14}$$

with G^j bounded below by M^j for $j = 1, \dots, m$. This friction term plays a role in the case of non-locked inclusions.

Problem $\mathcal{P}$. Given $f = (f^0, \dots, f^m)$ in $L^2(\Omega)$ find a minimizer over $\mathcal{K}$ of the functional

$$\mathcal{E}(v) \doteq \frac{1}{2}\mathbf{a}\big(e(v), e(v)\big) + \sum_{j=0}^{m} \Psi^j\big([v]_{S^j}\big) - \int_{\Omega^*} f v\,\mathrm{d}x. \tag{15}$$

From standard arguments, Problem $\mathcal{P}$ is equivalent to the variational inequality

Problem $\mathcal{P}'$. Find $u \in \mathcal{K}$ such that for every $v \in \mathcal{K}$,

$$\mathbf{a}\big(e(u), e(v-u)\big) + \sum_{j=0}^{m}\big(\Psi^j\big([v]_{S^j}\big) - \Psi^j\big([u]_{S^j}\big)\big) \geqslant \int_{\Omega^*} f(v-u)\,\mathrm{d}x. \tag{16}$$

We refer to Kikuchi–Oden [9, Section 10.3] for a detailed study of the Tresca conditions.

3.2. Statement of the main result

Let the number δ_j be defined for $j = 1, \dots, m$ as follows:

$$\delta_j = \begin{cases} 0 & \text{if } \Omega^j \text{ is a locked domain,} \\ 1 & \text{otherwise.} \end{cases} \tag{17}$$

For non-locked domains, we make an extra assumption on some moments of the right-hand side f^j.

Definition 3.1. For a spherical domain Ω^j, $\mathfrak{M}_j$ denotes the moment of f^j with respect to the center of Ω^j.
For a cylindrical domain Ω^j, $\mathfrak{M}_j$ denotes the moment of f^j with respect to the axis of Ω^j.

The extra hypothesis is

$$\big(\mathcal{H}^j\big): \quad \begin{cases} \text{For } \delta_j = 1 \text{ the moment } \mathfrak{M}_j \text{ is assumed to be “small”} \\ \text{compared to } M^j\,. \end{cases} \tag{18}$$

Proposition 3.2 *(Existence and uniqueness of the solution of $\mathcal{P}$). Let f be in $L^2(\Omega)$, g^j in $L^1(S^j)$ for $j = 1, \dots, m$. Assume that for $\delta^j = 1$ (non-locked domains), the function Ψ^j is coercive in the sense that the constant M^j of (13) is strictly positive and that hypothesis $(\mathcal{H}^j)$ holds. Then, there is a solution $u = (u^0, \dots, u^m)$ in $\mathcal{K}$ for Problem $\mathcal{P}'$. Uniqueness holds only for u^0 and for $u^j - \mathrm{proj}_{\mathcal{R}} u^j$, $j = 1, \dots, m$ (equivalently for $e(u^j)$).*

Remark 3.3. Proposition 3.2 is still true with $M^j = 0$ provided the corresponding $\mathfrak{M}_j = 0$ (only for $\delta_j = 1$). This covers the case of frictionless contact.
On the other hand, no condition is required on $\mathfrak{M}_j$ if the corresponding Ψ^j has superlinear growth.

In a joint forthcoming paper with Doina Cioranescu and Julia Orlik [1], the periodic homogenization of this problem is studied.

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