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Mathematical Problems in Mechanics

Long cycle behavior of the plastic deformation of an elasto-perfectly-plastic oscillator with noise ☆

Le comportement de la déformation plastique pour un oscillateur élastique-parfaitement-plastique excité par un bruit blanc

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ABSTRACT

For decades, a vast amount of research effort in experimental engineering together with numerical simulations has been devoted to the study of the plastic deformation and total deformation of elasto-perfectly-plastic (EPP) oscillators. All of these results reveal that both the plastic and total deformations of an EPP oscillator, being excited by a white noise, have variances that increase linearly with time and share a common asymptotic growth rate. Before our present work, there was no apparent theoretical justification on this empirical observation. In this paper, we use a stochastic variational inequality (SVI) for the modeling of the evolution between the velocity of an EPP oscillator and its non-linear restoring force; and this modeling has already been justified in some previous works of the authors. By introducing the novel notion of long cycle behavior of the Markovian solution of the corresponding SVI, we first establish a mathematical explanation for the empirical observation and characterize the mentioned asymptotic growth rate in terms of certain stopping times read off from the trajectory; secondly, we show an effective method on computing this asymptotic growth rate, which has been a long lasting challenging question to engineers. Finally numerical simulation is provided to illustrate the notable agreement between our theoretical prediction and empirical studies in the engineering literature.

RÉSUMÉ

Des résultats expérimentaux en sciences de l'ingénieur ont montré que, pour un oscillateur élasto-plastique-parfait excité par un bruit blanc, la déformation plastique et la déformation totale ont une variance, qui asymptotiquement, croît linéairement avec le temps avec le même coefficient. Dans ce travail, nous prouvons ce résultat et nous caractérisons le coefficient de dérive. Notre étude repose sur une inéquation variationnelle stochastique gouvernant l'évolution entre la vitesse de l'oscillateur et la force de rappel non-linéaire. Nous définissons alors le comportement en cycles longs du processus de Markov solution

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de l'inéquation variationnelle stochastique qui est le concept clé pour obtenir le résultat. Une question importante en sciences de l'ingénieur est de calculer ce coefficient. Les résultats numériques confirment avec succès notre prédiction théorique et les études empiriques faites par les ingénieurs.

Version française abrégée

Dans cet article, nous étudions la variance de l'oscillateur élastique-parfaitement-plastique (EPP) excité par un bruit blanc. La dynamique de l'oscillateur s'exprime à l'aide d'une équation à mémoire (voir (1)–(2)). A. Bensoussan et J. Turi [1] ont montré que la relation entre la vitesse et la composante élastique satisfait une inéquation variationnelle stochastique (voir ($\mathcal{SVI}$)). Dans ce cadre, nous introduisons les cycles long indépendants (définis plus loin) et nous justifions qu'ils permettent de caractériser la variance de la déformation plastique et de la déformation totale (voir (4)).

1. Introduction

In the civil engineering literature, elasto-perfectly-plastic (EPP) oscillators are frequently employed for the estimation of the critical time of failure of mechanical structures subject to random vibrations. This EPP oscillator is essentially a simple one-dimensional model, which is good enough to describe the elasto-plastic behavior of a board class of commonly found mechanical structures whose total deformations mainly response to their own first modal deformations.

The main difficulty of studying these systems comes from a high frequency of repeating visits to plastic phases in a small time interval in a recurrent manner. A plastic deformation is produced when the stress of the structure reaches beyond an elastic limit. The dynamics of the EPP-oscillator has memory, which is commonly described in the engineering literature by a process $x(t)$, *displacement of the oscillator*, that takes into account of the underlying hysteresis. We now interest in studying the problem

$$\ddot{x} + c_0\dot{x} + \mathbf{F}_t = \dot{w} \tag{1}$$

with initial conditions on its displacement and velocity given $x(0) = 0$ and $\dot{x}(0) = 0$ respectively. Here $c_0 > 0$ is the viscous damping coefficient, $k > 0$ is the stiffness, w is a Wiener process. The restoring force $\mathbf{F}_t \triangleq \mathbf{F}(x(s), 0 \leqslant s \leqslant t)$ is a non-linear functional which depends on the whole trajectory $\{x(s)\colon\ 0 \leqslant s \leqslant t\}$ up to time t. The plastic deformation denoted by $\Delta(t)$ at time t can be recovered from the pair $(x(t), \mathbf{F}_t)$ by using the following relationship:

$$\mathbf{F}_t = \begin{cases} kY & \text{if } x(t) = Y + \Delta(t), \\ k(x(t) - \Delta(t)) & \text{if } x(t) \in (-Y + \Delta(t), Y + \Delta(t)), \\ -kY & \text{if } x(t) = -Y + \Delta(t), \end{cases} \tag{2}$$

where $\Delta(t) = \int_0^t \mathbf{1}_{\{|\mathbf{F}_t| = kY\}}\,\mathrm{d}x(s)$ and Y is the elasto-plastic bound. Karnopp and Scharton [4] proposed a model with a separation between elastic and plastic phases. In particular, they introduced a fictitious variable (or called the elastic component) $z(t) \triangleq x(t) - \Delta(t)$, and also noticed the simple fact that during the transition period between two consecutive plastic phases, $z(t)$ behaves like a linear oscillator. In addition, both experimental and computational works had been done by engineers in [3], in which they revealed that total deformation has a variance that increases linearly with time:

$$\lim_{T\to\infty} \frac{\sigma^2(x(T))}{T} = \sigma^2, \tag{3}$$

where $\sigma^2 \in \mathbb{R}^+$.

In a recent work by the first author together with J. Turi in [1], a proper mathematical framework for modeling elasto-plastic oscillators with noise (to be described as below) has been formulated in terms of stochastic variational inequalities (SVI); indeed, that is the inequality governing the relationship between the velocity $y(t)$ and the elastic component $z(t)$:

$$\mathrm{d}y(t) = -\big(c_0 y(t) + kz(t)\big)\,\mathrm{d}t + \mathrm{d}w(t), \qquad \big(\mathrm{d}z(t) - y(t)\,\mathrm{d}t\big)\big(\phi - z(t)\big) \geqslant 0, \quad \forall |\phi| \leqslant Y,\ |z(t)| \leqslant Y. \tag{$\mathcal{SVI}$}$$

The plastic deformation $\Delta(t)$ can be recovered by $\int_0^t y(s)\mathbf{1}_{\{|z(s)|=Y\}}\,\mathrm{d}s$.

In this Note, our objective is to provide a mathematical justification on the limiting behavior in (3) together with its exact mathematical characterization based on ($\mathcal{SVI}$). We first introduce the notion of *long cycles* by identifying a sequence of independent components in the trajectory.

1.1. Long cycles

Denote $\tau_0 \triangleq \inf\{t > 0\colon\ y(t) = 0 \text{ and } |z(t)| = Y\}$, and $\delta \triangleq \operatorname{sign}(z(\tau_0))$ which labels the first hitting of $(y(t), z(t))$ to the boundary. Define $\theta_0 \triangleq \inf\{t > \tau_0\colon\ y(t) = 0 \text{ and } z(t) = -\delta Y\}$. Recursively, we also define, for each $n \geqslant 0$:

$$\tau_{n+1} \triangleq \inf\{t > \theta_n:\ y(t) = 0 \text{ and } z(t) = \delta Y\},$$
$$\theta_{n+1} \triangleq \inf\{t > \tau_{n+1}:\ y(t) = 0 \text{ and } z(t) = -\delta Y\}.$$

With these definitions of τ_n's and θ_n's in mind, the n-th long cycle is defined (resp. the first part of the cycle and the second part of the cycle) as the component of trajectory only defined on the interval $[\tau_n, \tau_{n+1})$ ($[\tau_n, \theta_{n+1})$ and $[\theta_{n+1}, \tau_{n+1})$). Due to its Markovian nature, at every instant τ_n (and beyond up to τ_{n+1}), the process $(y(t), z(t))$ restarts anew as if it were at τ_0 (and before τ_1). Moreover, there are two types of cycles depending on the sign of δ. The family of stopping times $\{\tau_n,\ n \geqslant 0\}$ represents the commencement times of long cycles. It is worth to remark that the plastic and total deformations are the same on a long cycle since

$$\int_{\tau_0}^{\tau_1} y(t)\,\mathrm{d}t = \int_{\tau_0}^{\tau_1} y(t)\mathbf{1}_{\{|z(s)|=Y\}}\,\mathrm{d}t + \int_{\tau_0}^{\tau_1} y(t)\mathbf{1}_{\{|z(s)|<Y\}}\,\mathrm{d}t$$

and that

$$\int_{\tau_0}^{\tau_1} y(t)\mathbf{1}_{\{|z(s)|<Y\}}\,\mathrm{d}t = z(\tau_1) - z(\tau_0) = 0.$$

The following theorem is the main result in this Note:

Theorem 1.1 *(Characterization of the variance related to the plastic/total deformation). Using the notations as above, we have the characterization,*

$$\lim_{T\to\infty} \frac{\sigma^2(x(T))}{T} = \frac{\mathbb{E}(\int_{\tau_0}^{\tau_1} y(t)\,\mathrm{d}t)^2}{\mathbb{E}(\tau_1 - \tau_0)}. \tag{4}$$

2. Supporting numerical evidence

In this section, we provide computational results of formula (4) which confirm our theoretical results. A **C** code has been written for simulating $(y(t), z(t))$ where, as explained in Section 4, without loss of generality we can suppose that $(y(0), z(0)) = (0, -Y)$ or $(0, Y)$ with equal probability 1/2. See [2] for the direct algorithmic numerical scheme. Let $T > 0$, $N \in \mathbb{N}$ and $\{t_n = n\delta t:\ 0 \leqslant n \leqslant N\}$ where $\delta t = \frac{T}{N}$. To compute the left-hand side of (4), we consider $MC \in \mathbb{N}$ and we generate MC numerical solutions of $(\mathcal{SVI})$ $\{y^i(t):\ 0 \leqslant t \leqslant T \text{ and } 1 \leqslant i \leqslant MC\}$ up to the time T. By the law of large numbers, we can approximate $\frac{1}{T}\mathbb{E}x(T)^2$ by

$$X_{MC} \triangleq \frac{1}{T}\frac{1}{MC}\sum_{i=1}^{MC}\left(\sum_{i=1}^{N} y^i(t_n)\delta t\right)^2$$

and also $\frac{1}{T^2}\mathbb{E}x(T)^4$ by

$$X^2_{MC} \triangleq \frac{1}{T^2}\frac{1}{MC}\sum_{i=1}^{MC}\left(\sum_{i=n}^{N} y^i(t_n)\delta t\right)^4.$$

Define $s_X \triangleq \sqrt{X^2_{MC} - (X_{MC})^2}$, we also know by the central limit theorem that

$$\frac{1}{T}\mathbb{E}x(T)^2 \in \left[X_{MC} - \frac{1.96 s_X}{\sqrt{MC}}, X_{MC} + \frac{1.96 s_X}{\sqrt{MC}}\right]$$

at the 95% confidence level. Similarly, to compute the right-hand side of (4), we generate MC numerical long cycles. For each trajectory $\{y^i(t),\ t \geqslant 0\}$, we consider N^i_c the required number of time step to obtain a completed cycle. Define

$$\delta_{MC} \triangleq \frac{1}{MC}\sum_{i=1}^{MC}\left(\sum_{n=0}^{N^i_c} y^i(t_n)\delta t\right)^2,$$
$$\delta^2_{MC} \triangleq \frac{1}{MC}\sum_{i=1}^{MC}\left(\sum_{n=0}^{N^i_c} y^i(t_n)\delta t\right)^4,$$

Table 1
Monte Carlo simulations to compare numerical solution of the left- and right-hand sides of (4), $T = 500$, $\delta t = 10^{-4}$ and $MC = 10\,000$.

$c_0 = 1$, $k = 1$				
Y	X_{MC}, $T = 500$	$\frac{\delta_{MC}}{\tau_{MC}}$	τ_{MC}	Relative error %
0.1	$0.807^{\pm 0.031}$	$0.815^{\pm 0.072}$	$6.61^{\pm 0.11}$	1.3
0.2	$0.649^{\pm 0.026}$	$0.616^{\pm 0.050}$	$8.74^{\pm 0.13}$	5.1
0.3	$0.493^{\pm 0.020}$	$0.470^{\pm 0.037}$	$10.45^{\pm 0.16}$	4.7
0.4	$0.361^{\pm 0.014}$	$0.353^{\pm 0.028}$	$12.12^{\pm 0.18}$	2.2
0.5	$0.266^{\pm 0.011}$	$0.257^{\pm 0.020}$	$13.80^{\pm 0.21}$	3.4

$$\tau_{MC} \triangleq \frac{1}{MC}\sum_{i=1}^{MC} N_c^i \delta t,$$

$$\tau_{MC}^2 \triangleq \frac{1}{MC}\sum_{i=1}^{MC} \left(N_c^i \delta t\right)^2,$$

$$s_\delta \triangleq \sqrt{\delta_{MC}^2 - (\delta_{MC})^2} \quad \text{and}$$

$$s_\tau \triangleq \sqrt{\tau_{MC}^2 - (\tau_{MC})^2}.$$

We also know that

$$\frac{\delta_{MC}}{\tau_{MC}} \in \left[\frac{\delta_{MC} - \frac{3s_\delta}{\sqrt{MC}}}{\tau_{MC} + \frac{3s_\tau}{\sqrt{MC}}}, \frac{\delta_{MC} + \frac{3s_\delta}{\sqrt{MC}}}{\tau_{MC} - \frac{3s_\tau}{\sqrt{MC}}}\right]$$

at the 95% confidence level. In Table 1, we illustrate a comparison of the results obtained for X_{MC} and $\frac{\delta_{MC}}{\tau_{MC}}$ where $T = 500$, $\delta t = 10^{-4}$ and $MC = 10\,000$.

Remark 2.1. From a point of view of numerical simulation, the behavior in long cycles is critical and crucial. Indeed, the computation of the left-hand side of (4) is much more time-consuming in comparison with that of the right-hand side (see the τ_{MC}-column of Table 1).

Our proof relies on solving two non-local partial differential equations related to long cycles.

3. The issue of long cycles and plastic deformations

Notation 1. $D \triangleq \mathbb{R} \times (-Y, +Y)$, $D^+ \triangleq (0, \infty) \times \{Y\}$, $D^- \triangleq (-\infty, 0) \times \{-Y\}$, and the differential operators $A\zeta \triangleq -\frac{1}{2}\zeta_{yy} + (c_0 y + kz)\zeta_y - y\zeta_z$, $B_+\zeta \triangleq -\frac{1}{2}\zeta_{yy} + (c_0 y + kY)\zeta_y$, $B_-\zeta \triangleq -\frac{1}{2}\zeta_{yy} + (c_0 y - kY)\zeta_y$, where ζ is a smooth enough function on D.

Let f be a bounded measurable function, we want to solve for $v^+(f)$ and $v^-(f)$ such that

$$Av^+(f) = f(y,z) \quad \text{in } D, \qquad B_+v^+(f) = f(y,Y) \quad \text{in } D^+, \qquad B_-v^+(f) = f(y,-Y) \quad \text{in } D^-, \tag{P_{v^+}}$$

with the non-local boundary conditions $v^+(f)(y,Y)$ being continuous in y and $v^+(f)(0^-,-Y) = 0$, and

$$Av^-(f) = f(y,z) \quad \text{in } D, \qquad B_+v^-(f) = f(y,Y) \quad \text{in } D^+, \qquad B_-v^-(f) = f(y,-Y) \quad \text{in } D^-, \tag{P_{v^-}}$$

with the non-local boundary conditions $v^-(f)(0^+,Y) = 0$ and $v^-(f)(y,-Y)$ being continuous in y. According to the Feynman–Kac formula, the functionals

$$v^+(f) = \mathbb{E}\left(\int_{\tau_0}^{\theta_0} f\big(y(s), z(s)\big)\, ds \,|\, z(\tau_0) = Y\right)$$

and

$$v^-(f) = \mathbb{E}\left(\int_{\tau_0}^{\theta_0} f\big(y(s), z(s)\big)\, ds \,|\, z(\tau_0) = -Y\right);$$

and functionally, they are regarded as *half long cycles* in the operator sense. In addition, we define $\pi^+(y,z)$ and $\pi^-(y,z)$ by

$$A\pi^+ = 0 \quad \text{in } D, \qquad \pi^+(y,Y) = 1 \quad \text{in } D^+, \qquad \pi^+(y,-Y) = 0 \quad \text{in } D^-, \tag{P_{π^+}}$$

and

$$A\pi^{-}=0 \quad \text{in } D, \qquad \pi^{-}(y,Y)=0 \quad \text{in } D^{+}, \qquad \pi^{-}(y,-Y)=1 \quad \text{in } D^{-}. \tag{P_{π^-}}$$

Note that $\pi^{+}+\pi^{-}=1$, so the existence and uniqueness of a bounded solution (P_{π^+}) and (P_{π^-}) are clear.

Proposition 3.1. *$\pi^{+}(0^{+},-Y)>0$ and $\pi^{-}(0^{-},Y)>0$.*

Proof. We only check the first inequality, the second can be shown similarly. Without loss of generality, we assume that $4k>c_0^2$, and we now consider the elastic process $(y_{yz}(t), z_{yz}(t))$:

$$z_{yz}(t)=e^{\frac{-c_0 t}{2}}\left\{z\cos(\omega t)+\frac{1}{\omega}\left(y+\frac{c_0}{2}z\right)\sin(\omega t)\right\}+\frac{1}{\omega}\int_0^t e^{-\frac{c_0}{2}(t-s)}\sin\bigl(\omega(t-s)\bigr)\,\mathrm{d}w(s),$$

$$y_{yz}(t)=-\frac{c_0}{2}z_{yz}(t)+e^{-\frac{c_0 t}{2}}\left\{-\omega z\sin(\omega t)+\left(y+\frac{c_0}{2}z\right)\cos(\omega t)\right\}+\int_0^t e^{-\frac{c_0}{2}(t-s)}\cos\bigl(\omega(t-s)\bigr)\,\mathrm{d}w(s)$$

where $\omega\triangleq\frac{\sqrt{4k-c_0^2}}{2}$. Set $\tau_{yz}\triangleq\inf\{t>0\colon\ |z_{yz}(t)|\geqslant Y\}$, we have $\pi^{+}(y,z)=\mathbb{P}(z_{yz}(\tau_{yz})=Y)$ and $\pi^{-}(y,z)=\mathbb{P}(z_{yz}(\tau_{yz})=-Y)$. We next claim that

$$\pi^{-}(y,z)\to 1 \quad \text{as } y\to-\infty,\ z\in[-Y,Y]. \tag{5}$$

Indeed, $\forall t$ with $0<t<\frac{\pi}{\omega}$ we have $z_{yz}(t)\to-\infty$, as $y\to-\infty$ a.s. Therefore $\forall t$ with $0<t<\frac{\pi}{\omega}$, $z_{yz}(t)<-Y$ a.s. for $-y$ sufficiently large. Hence, $\tau_{yz}<t$ a.s. for $-y$ sufficiently large. Therefore a.s. $\limsup_{y\to-\infty}\tau_{yz}<t$. Since t is arbitrary, necessarily a.s. $\tau_{yz}\to 0$, as $y\to-\infty$ which implies (5). Moreover, $\pi^{-}(y,Y)$ cannot have a minimum or a maximum at any finite $y<0$; it is then monotone and since $\pi^{-}(-\infty,Y)=1$, it is monotonic decreasing. It follows that $\pi^{-}(0^{-},Y)<1$, which cannot be 0. Otherwise, $\pi^{-}(y,Y)$ is continuous at $y=0$, and $(0,Y)$ is a point of minimum of $\pi^{-}(y,z)$ and we must have $\pi_{yy}(0^{-},Y)>0$. Since for $y<0$, $\pi_y^{-}(y,Y)<0$, so from the equation of π^{-} we get $\limsup_{y\to0^{-}}\pi_{yy}^{-}(y,Y)\leqslant 0$ which is not possible since $(0,Y)$ is a minimum. $\square$

We next define $\eta(y,z)$ by

$$A\eta=f(y,z) \quad \text{in } D, \qquad \eta(y,Y)=0 \quad \text{in } D^{+}, \qquad \eta(y,-Y)=0 \quad \text{in } D^{-} \tag{P_η}$$

with the local boundary conditions $\eta(0^{+},Y)=0$ and $\eta(0^{-},-Y)=0$. For any bounded measurable f, problem (P_η) attains a unique bounded measurable solution. Define $\varphi_{+}(y;f)$ as the solution of the following:

$$-\frac{1}{2}\varphi_{+,yy}+(c_0 y+kY)\varphi_{+,y}=f(y,Y), \quad y>0,\ \varphi_{+}(0;f)=0. \tag{6}$$

One can apply integrating factors and conclude that

$$\varphi_{+}(y;f)=2\int_0^\infty \mathrm{d}\xi\,\exp\bigl(-\bigl(c_0\xi^2+2kY\xi\bigr)\bigr)\int_\xi^{\xi+y} f(\zeta;Y)\exp\bigl(-2c_0\xi(\zeta-\xi)\bigr)\,\mathrm{d}\zeta, \quad y\geqslant 0. \tag{7}$$

Also define $\psi_{+}(y,z;f)$ by:

$$A\psi_{+}=0, \quad \text{in } D, \qquad \psi_{+}(y,Y)=\varphi_{+}(y;f), \quad \text{in } D^{+}, \qquad \psi_{+}(y,-Y)=0, \quad \text{in } D^{-}. \tag{P_{ψ_+}}$$

Similarly, we also define $\varphi_{-}(y;f)$ and $\psi_{-}(y,z;f)$ by

$$-\frac{1}{2}\varphi_{-,yy}+(c_0 y-kY)\varphi_{-,y}=f(y,-Y), \quad y<0,\ \varphi_{-}(0;f)=0, \tag{8}$$

which leads to

$$\varphi_{-}(y;f)=2\int_0^\infty \mathrm{d}\xi\ \exp\bigl(-\bigl(c_0\xi^2-2kY\xi\bigr)\bigr)\int_{y-\xi}^{-\xi} f(\zeta;-Y)\exp\bigl(-2c_0\xi(\zeta-\xi)\bigr)\,\mathrm{d}\zeta, \quad y\leqslant 0,$$

and

$$A\psi_{-}=0 \quad \text{in } D, \qquad \psi_{-}(y,Y)=0 \quad \text{in } D^{+}, \qquad \psi_{-}(y,-Y)=\varphi_{-}(y;f) \quad \text{in } D^{-}. \tag{P_{ψ_-}}$$

We can state the following proposition:

Proposition 3.2. *The solutions of* (P_{v^+}) *and* (P_{v^-}) *are given by*

$$v^+(f)(y,z) = \eta(y,z;f) + \psi_+(y,z;f) + \psi_-(y,z;f) + \frac{\eta(0^-,Y;f) + \psi_+(0^-,Y;f) + \psi_-(0^-,Y;f)}{\pi^-(0^-,Y)}\pi^+(y,z)$$

and

$$\begin{aligned} v^-(f)(y,z) &= \eta(y,z;f) + \psi_+(y,z;f) + \psi_-(y,z;f) \\ &\quad + \frac{\eta(0^+,-Y;f) + \psi_+(0^+,-Y;f) + \psi_-(0^+,-Y;f)}{\pi^+(0^-,Y)}\pi^-(y,z). \end{aligned}$$

Proof. Direct checking. □

4. Complete cycle

Firstly, let us check that $\mathbb{E}[x(t)] = 0$ and hence we simply have $\sigma^2(x(t)) = \mathbb{E}[x^2(t)]$. Indeed, upon the symmetry of the underlying ($\mathcal{SVI}$) and the choice of initial conditions $y(0) = 0$, $z(0) = 0$, the processes $(y(t), z(t))$ and $-(y(t), z(t))$ possess the same law. Then,

$$\mathbb{E}\big[x(t)\big] = \mathbb{E}\left[\int_0^t y(s)\,\mathrm{d}s\right] = -\mathbb{E}\left[\int_0^t y(s)\,\mathrm{d}s\right] = 0.$$

In addition, $(y(\tau_1), z(\tau_1))$ is equal to $(0, -Y)$ or $(0, Y)$ with equal probability $1/2$; accordingly, without loss of generality, we suppose that $(y(0), z(0)) = (0, -Y)$ or $(0, Y)$ with equal probability $1/2$. Let us treat the case that $y(0) = 0$ and $z(0) = Y$. So, $\tau_0 = 0$ and $\theta_1 = \inf\{t > 0\colon\ z(t) = -Y \text{ and } y(t) = 0\}$. We can assert that $\mathbb{E}\theta_1 = v^+(1)(0, Y)$ hence $\theta_1 < \infty$ a.s. since η is bounded. Define $\tau_1 = \inf\{t > \theta_1\colon\ z(t) = Y \text{ and } y(t) = 0\}$, then

$$\mathbb{E}\tau_1 = v^+(1)(0,Y) + v^-(1)(0,-Y) = 2v^+(1)(0,Y)$$

by symmetry. At time τ_1 the state of the system is again $(0, Y)$. So the sequence $\{\tau_n\colon\ n \geqslant 0\}$ is such that $\tau_n < \tau_{n+1}$ and in the interval (τ_n, τ_{n+1}) we have a cycle probabilistically identical to that in $(0, \tau_1)$. Consider the random integral $\int_0^{\tau_1} y(t)\,\mathrm{d}t$. We have

$$\mathbb{E}\int_0^{\tau_1} y(t)\,\mathrm{d}t = \mathbb{E}\int_0^{\theta_1} y(t)\,\mathrm{d}t + \mathbb{E}\int_{\theta_1}^{\tau_1} y(t)\,\mathrm{d}t = v^+(y)(0,Y) + v^-(y)(0,-Y).$$

On the other hand, for any antisymmetric f, i.e. $f(-y,-z) = -f(y,z)$, we have $v^+(f)(0,Y) = -v^-(f)(0,-Y)$; in particular, y is antisymmetric, and so $\mathbb{E}\int_0^{\tau_1} y(t)\,\mathrm{d}t = 0$. Now, upon mutual independence of components over disjoint time intervals,

$$\mathbb{E}\left(\int_0^{\tau_n} y(t)\,\mathrm{d}t\right)^2 = \mathbb{E}\left(\sum_{j=0}^{n-1}\int_{\tau_j}^{\tau_{j+1}} y(t)\,\mathrm{d}t\right)^2 = n\mathbb{E}\left(\int_0^{\tau_1} y(t)\,\mathrm{d}t\right)^2.$$

Then

$$\frac{\mathbb{E}(\int_0^{\tau_n} y(t)\,\mathrm{d}t)^2}{\mathbb{E}\tau_n} = \frac{\mathbb{E}(\int_0^{\tau_1} y(t)\,\mathrm{d}t)^2}{\mathbb{E}\tau_1}.$$

Let $T > 0$ and N_T with $\tau_{N_T} \leqslant T < \tau_{N_T+1}$, $N_T = 0$ if $\tau_1 > T$, and define $\tau_0 = 0$ and further calculations lead to the following

$$\frac{\mathbb{E}(\int_0^{\tau_{N_T+1}} y(t)\,\mathrm{d}t)^2}{\mathbb{E}\tau_{N_T+1}} = \frac{\mathbb{E}(\int_0^{\tau_1} y(t)\,\mathrm{d}t)^2}{\mathbb{E}\tau_1}.$$

Next, we can justify that

$$\lim_{T\to\infty}\frac{1}{T}\mathbb{E}\left(\int_0^T y(t)\,\mathrm{d}t\right)^2 = \lim_{T\to\infty}\frac{1}{T}\mathbb{E}\left(\int_0^{\tau_{N_T+1}} y(t)\,\mathrm{d}t\right)^2, \tag{9}$$

and that we have a lower bound and an upper bound for the right-hand side of (9), that is

$$\frac{\mathbb{E}(\int_0^{\tau_1} y(t)\,\mathrm{d}t)^2}{\mathbb{E}\tau_1} \leqslant \frac{1}{T}\mathbb{E}\left(\int_0^{\tau_{N_T+1}} y(t)\,\mathrm{d}t\right)^2 \leqslant \left(1+\frac{\mathbb{E}\tau_1}{T}\right)\frac{\mathbb{E}(\int_0^{\tau_1} y(t)\,\mathrm{d}t)^2}{\mathbb{E}\tau_1}.$$

Therefore $\lim_{T\to\infty}\frac{1}{T}\mathbb{E}(\int_0^T y(t)\,\mathrm{d}t)^2 = \frac{\mathbb{E}(\int_0^{\tau_1} y(t)\,\mathrm{d}t)^2}{\mathbb{E}\tau_1}$. The proof is done.

5. Conclusion

In this Note, we have introduced a probabilistic formula for the asymptotic growth rate of the variance of the total (or plastic) deformation related to an elasto-perfectly-plastic oscillator with noise. Moreover, our formula allows fast probabilistic simulations to compute this asymptotic rate since the mean time durations of long cycle are very short (see the third column of Table 1).

In our next work, we will provide a purely analytic formula for the term involving a square in the expectation at the numerator in the right hand of (4). Indeed, there is no direct PDE representation. Then, we will overcome this difficulty by proposing a non-local PDE describing the Fourier transform of the plastic deformation.

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