C. R. Acad. Sci. Paris, Ser. I 350 (2012) 921–924

Contents lists available at SciVerse ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Topology

On a conjecture of Dunfield, Friedl and Jackson

Sur une conjecture de Dunfield, Friedl et Jackson

Takayuki Morifuji

Department of Mathematics, Hiyoshi Campus, Keio University, Yokohama 223-8521, Japan

ARTICLE INFO

Article history:
Received 19 September 2012
Accepted after revision 10 October 2012
Available online 25 October 2012

Presented by the Editorial Board

ABSTRACT

In this short Note, we show that the twisted Alexander polynomial associated to a parabolic $SL(2,\mathbb{C})$-representation detects genus and fibering of the twist knots. As a corollary, a conjecture of Dunfield, Friedl and Jackson is proved for the hyperbolic twist knots.

RÉSUMÉ

Dans cette courte Note, nous montrons que le polynome Alexander tordu a associe a une representation parabolique detecte genre et fibering des noeuds de la torsion. Comme un corollaire, une conjecture de Dunfield, Friedl et Jackson est prouvee pour les noeuds de la torsion hyperboliques.

1. Introduction

Let K be a knot in the 3-sphere S^3 and denote its knot group by $G(K)$. That is, $G(K)=\pi_1 E(K)$ where $E(K)$ is the knot exterior $S^3\setminus\mathrm{int}(N(K))$ which is a compact 3-manifold with torus boundary. For a nonabelian representation $\rho: G(K)\to SL(2,\mathbb{C})$, the *twisted Alexander polynomial* $\Delta_{K,\rho}(t)\in\mathbb{C}[t^{\pm1}]$ is defined up to multiplication by some t^i, with $i\in\mathbb{Z}$, see [10], [16] and [9] for details.

If K is a hyperbolic knot, namely the interior of $E(K)$ admits the complete hyperbolic metric with finite volume, there is a discrete faithful representation $\rho_0: G(K)\to SL(2,\mathbb{C})$, which is called the *holonomy representation*, corresponding to the hyperbolic structure.

The *hyperbolic torsion polynomial* $\mathcal{T}_K(t)\in\mathbb{C}[t^{\pm1}]$ was defined in [3] for hyperbolic knots as a suitable normalization of $\Delta_{K,\rho_0}(t)$. It is a symmetric polynomial in the sense that $\mathcal{T}_K(t^{-1})=\mathcal{T}_K(t)$, which seems to contain geometric information. In fact Dunfield, Friedl and Jackson conjectured in [3] that $\mathcal{T}_K$ determines the genus $g(K)$ and moreover, the knot K is fibered if and only if $\mathcal{T}_K$ is monic.

They show in [3] that the conjecture holds for all hyperbolic knots with at most 15 crossings. Our main theorem in this note is the following:

Theorem 1.1. *For all hyperbolic twist knots K (see Fig. 1) $\mathcal{T}_K$ determines the genus $g(K)$ and moreover, the knot K is fibered if and only if $\mathcal{T}_K$ is monic.*

E-mail address: morifuji@z8.keio.jp.

1631-073X/$ – see front matter
http://dx.doi.org/10.1016/j.crma.2012.10.013

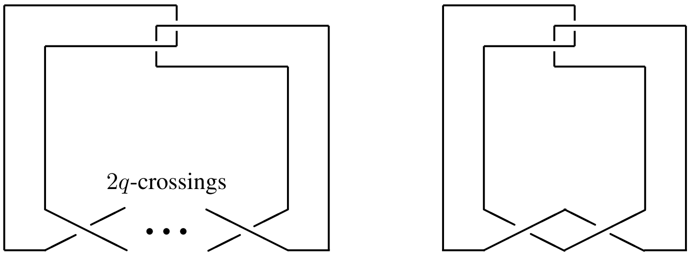

Fig. 1. $K = J(2, 2q)$ and the figure eight knot $J(2, -2)$.

Fig. 1. $K = J(2, 2q)$ et le chiffre huit noeud $J(2, -2)$.

As far as we know, this is the first infinite family of knots for which the conjecture is verified. Since twist knots are 2-bridge knots (in particular alternating knots), their genus and fibering can be detected by the Alexander polynomial (see [2,12–14]). However there seems to be no a priori reason that the same must be true for $\mathcal{T}_K$. See [3, Section 7], [8] for twisted Alexander polynomials and character varieties of knot groups.

Recall that an $SL(2, \mathbb{C})$-representation ρ is called *parabolic* if the meridian of $G(K)$ is sent to a parabolic element of $SL(2, \mathbb{C})$ and $\rho(G(K))$ is nonabelian. Since the holonomy representation of hyperbolic knots is parabolic, the above theorem is an immediate consequence of the following:

Theorem 1.2. *Let K be a twist knot and $\rho : G(K) \to SL(2, \mathbb{C})$ a parabolic representation. Then $\Delta_{K,\rho}(t)$ determines $g(K)$. Moreover K is fibered if and only if $\Delta_{K,\rho}(t)$ is monic.*

In the next section, we quickly review twisted Alexander polynomials of twist knots for parabolic $SL(2, \mathbb{C})$-representations (see [11, Sections 3, 4] for details). The proof of Theorem 1.2 will be given in Section 3.

2. Twisted Alexander polynomials of twist knots

Let $K = J(\pm 2, p)$ be the twist knot ($p \in \mathbb{Z}$). It is known that $J(\pm 2, 2q + 1)$ is equivalent to $J(\mp 2, 2q)$ and $J(\pm 2, p)$ is the mirror image of $J(\mp 2, -p)$. Hence we only consider the case where $K = J(2, 2q)$ for $q \in \mathbb{Z}$ (see Fig. 1). The knot $J(2, 0)$ presents the trivial knot, so that we always assume $q \neq 0$. The typical examples are the trefoil knot $J(2, 2)$ and the figure eight knot $J(2, -2)$.

The twist knots are alternating knots and have genus one. The Alexander polynomial of $K = J(2, 2q)$ is given by $\Delta_K(t) = q - (2q - 1)t + qt^2$. Furthermore, it is known (see [14]) that $J(2, 2q)$ is fibered if and only if $|q| = 1$. It is also known that $J(2, 2q)$ is hyperbolic if $q \notin \{0, 1\}$.

The knot group $G(J(2, 2q))$ has the presentation:

$$G\big(J(2, 2q)\big) = \big\langle x, y \bigm| w^q x = y w^q \big\rangle,$$

where $w = [y, x^{-1}]$. Suppose that $\rho : G(J(2, 2q)) \to SL(2, \mathbb{C})$ is a parabolic representation. After conjugating, if necessary, we may assume that

$$\rho(x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \rho(y) = \begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix}.$$

Let $\rho(w^q) = (a_{ij}(u))$ and write $\phi_q(u) = a_{11}(u)$. It is known that ρ defines a group representation when u satisfies $\phi_q(u) = 0$ (see [15, Theorem 2]). We call $\phi_q(u)$ the *Riley polynomial* of the twist knot $J(2, 2q)$. By [11, Proposition 3.1], $\phi_q(u)$ has an explicit formula

$$\phi_q(u) = (1 - u)\frac{\lambda_+^q - \lambda_-^q}{\lambda_+ - \lambda_-} - \frac{\lambda_+^{q-1} - \lambda_-^{q-1}}{\lambda_+ - \lambda_-},$$

where

$$\lambda_\pm(u) = \frac{u^2 + 2 \pm \sqrt{u^4 + 4u^2}}{2}$$

denote the eigenvalues of the matrix $\rho(w)$. Of course, the holonomy representation ρ_0 corresponds to one of the roots of $\phi_q(u) = 0$.

Lemma 2.1. *The Riley polynomial $\phi_q(u)$ satisfies the following*:

(1) *The highest coefficient of $\phi_q(u)$ is ± 1.*
(2) *$\phi_q(u) \in \mathbb{Z}[u]$ is irreducible.*
(3) *$\deg \phi_q(u) = 2q - 1$ $(q > 0)$ or $2|q|$ $(q < 0)$.*

Proof. (1) See [15, Theorem 2]. (2), (3) See [7, Theorem 1]. □

Example 2.2. We can easily check that $\phi_1(u) = 1 - u$, $\phi_{-1}(u) = 1 + u + u^2$, $\phi_2(u) = 1 - 2u + u^2 - u^3$, $\phi_{-2}(u) = 1 + 2u + 3u^2 + u^3 + u^4$ and $\phi_3(u) = 1 - 3u + 3u^2 - 4u^3 + u^4 - u^5$.

Lemma 2.3. *For a parabolic representation $\rho : G(K) \to SL(2, \mathbb{C})$ of $K = J(2, 2q)$, the twisted Alexander polynomial $\Delta_{K,\rho}(t)$ is given by*

$$\Delta_{K,\rho}(t) = \alpha\beta + \left\{\alpha + \beta - 2\alpha\beta + \frac{\lambda_+ - \lambda_-}{2 + \lambda_+ + \lambda_-}(\alpha - \beta)\right\}t + \alpha\beta t^2,$$

where $\alpha = 1 + \lambda_+ + \lambda_+^2 + \cdots + \lambda_+^{q-1}$ and $\beta = 1 + \lambda_- + \lambda_-^2 + \cdots + \lambda_-^{q-1}$.

Proof. We only have to put $s = 1$ in the formula of [11, Theorem 4.1]. □

Example 2.4. For $K = J(2, 2)$, there is just one parabolic representation up to conjugation and we have $\Delta_{K,\rho}(t) = 1 + t^2$. Similarly we obtain $\Delta_{K,\rho}(t) = 1 - 4t + t^2$ for any parabolic representation of $K = J(2, -2)$.

In general, the degree of the twisted Alexander polynomial gives a lower bound for the knot genus $g(K)$. In fact, for every nonabelian representation $\rho : G(K) \to SL(2, \mathbb{C})$, the following inequality holds (see [4]):

$$4g(K) - 2 \geqslant \deg \Delta_{K,\rho}(t). \tag{1}$$

When the equality holds in (1), we say $\Delta_{K,\rho}(t)$ determines the knot genus. For a fibered knot K, it is known that $\Delta_{K,\rho}(t)$ determines $g(K)$ and is a monic polynomial (see [1,4–6,9]).

3. Proof of Theorem 1.2

First we denote the highest coefficient of $\Delta_{K,\rho}(t)$ in Lemma 2.3 by $\gamma_q(u)$, namely $\gamma_q(u) = \alpha\beta$. Moreover we put $\tau_q(u) = \operatorname{tr}\rho(w^q) = \lambda_+^q + \lambda_-^q$. By [11, Corollary 4.3], $\tau_q(u) = \tau_{-q}(u)$ is a monic polynomial in $\mathbb{Z}[u]$ and $\deg \tau_q(u) = 2|q|$.

Example 3.1. Since $\tau_1(u) = u^2 + 2$, we obtain $\tau_{\pm 2}(u) = {\tau_1}^2 - 2 = u^4 + 4u^2 + 2$ and $\tau_{\pm 3}(u) = {\tau_1}^3 - 3\tau_1 = u^6 + 6u^4 + 9u^2 + 2$.

Now an easy calculation shows that

$$\begin{aligned}\gamma_q(u) &= \left(1 + \lambda_+ + \lambda_+^2 + \cdots + \lambda_+^{q-1}\right)\left(1 + \lambda_- + \lambda_-^2 + \cdots + \lambda_-^{q-1}\right) \\ &= \tau_{q-1}(u) + (\text{some polynomial in } \tau_1, \ldots, \tau_{q-2}).\end{aligned}$$

Thus we have $\deg \gamma_q(u) = 2|q| - 2$. By Lemma 2.1 (1), (2), if $\gamma_q(u) = 0$ for a complex number u satisfying $\phi_q(u) = 0$, then the Riley polynomial $\phi_q(u)$ divides $\gamma_q(u)$. But this contradicts the fact that

$$\begin{aligned}\deg \phi_q(u) &= 2|q| - \max\{\operatorname{sign}(q), 0\} \\ &> 2|q| - 2 = \deg \gamma_q(u).\end{aligned}$$

Hence $\gamma_q(u)$ never vanishes for the parabolic representations. Thus $\deg \Delta_{K,\rho}(t) = 2$ and hence it determines the genus.

A similar argument applied to $\gamma_q(u) - 1$ shows that $\Delta_{K,\rho}(t)$ is not a monic polynomial for the nonfibered twist knot $K = J(2, 2q)$ with $|q| > 1$. This completes the proof of Theorem 1.2.

Remark 3.2. For the 3830 nonfibered 2-bridge knots $K(a, b)$ with $b < a \leqslant 287$, Dunfield, Friedl and Jackson numerically compute the twisted Alexander polynomials for the parabolic representations. In fact, it is shown in [3, Section 7.6] that $\Delta_{K,\rho}(t)$ is nonmonic and determines the knot genus in every case.

Remark 3.3. Let $\delta_q(u)$ be the second coefficient of $\Delta_{K,\rho}(t)$ in Lemma 2.3. As we saw in Example 2.4, $\delta_{\pm 1}(u)$ are integers for the fibered twist knots $J(2, \pm 2)$. It is not so hard to show that $\delta_q(u) \in \mathbb{Z}[u]$ and $\deg \delta_q(u) = 2q - 4$ for $q > 1$. Therefore we can conclude that $\delta_q(u)$ with $q > 2$ is not a rational number for the parabolic representations. On the other hand, we have $\Delta_{K,\rho}(t) = (u^2 + 4) - 4t + (u^2 + 4)t^2$ for the hyperbolic twist knot $K = J(2, 4)$. In particular, we see that the second coefficient $\delta_2(u)$ is an integer for the holonomy representation, although $J(2, 4)$ is nonfibered (see [3, Section 6.5]).

Acknowledgements

The author would like to thank Stefan Friedl, Teruaki Kitano, Takahiro Kitayama and Yoshikazu Yamaguchi for helpful comments. The author also would like to thank the referee for useful comments, which helped to improve the Note. This research was partially supported by Grant-in-Aid for Scientific Research (No. 23540076), the Ministry of Education, Culture, Sports, Science and Technology, Japan.

References

[1] J.C. Cha, Fibred knots and twisted Alexander invariants, Trans. Amer. Math. Soc. 355 (2003) 4187–4200.
[2] R. Crowell, Genus of alternating link types, Ann. of Math. 69 (2) (1959) 258–275.
[3] N. Dunfield, S. Friedl, N. Jackson, Twisted Alexander polynomials of hyperbolic knots, Experiment. Math., in press, arXiv:1108.3045.
[4] S. Friedl, T. Kim, The Thurston norm, fibered manifolds and twisted Alexander polynomials, Topology 45 (2006) 929–953.
[5] S. Friedl, S. Vidussi, Twisted Alexander polynomials detect fibered 3-manifolds, Ann. of Math. 173 (2011) 1587–1643.
[6] H. Goda, T. Kitano, T. Morifuji, Reidemeister torsion, twisted Alexander polynomial and fibered knots, Comment. Math. Helv. 80 (2005) 51–61.
[7] J. Hoste, P.D. Shanahan, Trace fields of twist knots, J. Knot Theory Ramifications 10 (2001) 625–639.
[8] T. Kim, T. Morifuji, Twisted Alexander polynomials and character varieties of 2-bridge knot groups, Internat. J. Math. 23 (2012) 1250022, 24 pp.
[9] T. Kitano, T. Morifuji, Divisibility of twisted Alexander polynomials and fibered knots, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 4 (2005) 179–186.
[10] X.S. Lin, Representations of knot groups and twisted Alexander polynomials, Acta Math. Sin. (Engl. Ser.) 17 (2001) 361–380.
[11] T. Morifuji, Twisted Alexander polynomials of twist knots for nonabelian representations, Bull. Sci. Math. 132 (2008) 439–453.
[12] K. Murasugi, On the genus of the alternating knot I, J. Math. Soc. Japan 10 (1958) 94–105.
[13] K. Murasugi, On the genus of the alternating knot II, J. Math. Soc. Japan 10 (1958) 235–248.
[14] K. Murasugi, On a certain subgroup of the group of an alternating link, Amer. J. Math. 85 (1963) 544–550.
[15] R. Riley, Parabolic representations of knot groups, I, Proc. London Math. Soc. (3) 24 (1972) 217–242.
[16] M. Wada, Twisted Alexander polynomial for finitely presentable groups, Topology 33 (1994) 241–256.