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Number Theory

On the Erdős–Turán conjecture ☆

Sur la conjecture d'Erdös–Turán

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ABSTRACT

Let $\mathbb{N}$ be the set of all nonnegative integers. For a set $A \subseteq \mathbb{N}$, let $R(A,n)$ denote the number of solutions (a,a') of $a+a'=n$ with $a,a' \in A$. The well known Erdős–Turán conjecture says that if $R(A,n) \geqslant 1$ for all integers $n \geqslant 0$, then $R(A,n)$ is unbounded. In this Note, the following result is proved: There is a set $A \subseteq \mathbb{N}$ such that $R(A,n) \geqslant 1$ for all integers $n \geqslant 0$ and the set of n with $R(A,n)=2$ has density one.

RÉSUMÉ

Soit $\mathbb{N}$ l'ensemble des entiers positifs ou nul. Pour un sous-ensemble $A \subset \mathbb{N}$ nous notons $R(A,n)$ le nombre de solutions $(a,a') \in A^2$ de $a+a'=n$. La célèbre conjecture d'Erdös–Turán affirme que si $R(A,n) \geqslant 1$ pour tout entier $n \geqslant 0$, alors $R(A,n)$ n'est pas borné. Nous montrons dans cette Note qu'il existe un sous-ensemble $A \subset \mathbb{N}$ tel que $R(A,n) \geqslant 1$ pour tout entier $n \geqslant 0$ et tel que l'ensemble des n satisfaisant $R(A,n)=2$ soit de densité un.

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1. Introduction

Let $\mathbb{N}$ be the set of all nonnegative integers. For a set $A \subseteq \mathbb{N}$, let $R(A,n)$ denote the number of solutions (a,a') of $a+a'=n$ with $a,a' \in A$. If $R(A,n) \geqslant 1$ for all $n \in \mathbb{N}$, then A is called a *basis* of $\mathbb{N}$. The well known Erdős–Turán conjecture [4] says that if A is a basis of $\mathbb{N}$, then $R(A,n)$ is unbounded. Grekos, Haddad, Helou, and Pihko [5] proved that if A is a basis of $\mathbb{N}$, then $R(A,n) \geqslant 6$ for infinitely many positive integers n. Borwein, Choi, and Chu [1] improved 6 to 8. Nathanson [8] proved that the Erdős–Turán conjecture does not hold in $\mathbb{Z}$. For a set $A \subseteq \mathbb{Z}_m$, let $R_m(A,n)$ denote the number of solutions (a,a') of $a+a'=n$ with $a,a' \in A$. Developing Ruzsa's method [9], Tang and Chen [11] proved that for every sufficiently large integer m, there exists $A \subseteq \mathbb{Z}_m$ such that $1 \leqslant R_m(A,n) \leqslant 768$ for all $n \in \mathbb{Z}_m$. In 2008, Chen [2] proved that for every positive integer m, there exists $A \subseteq \mathbb{Z}_m$ such that $1 \leqslant R_m(A,n) \leqslant 288$ for all $n \in \mathbb{Z}_m$. In 1990, Ruzsa [9] found a subset A of $\mathbb{N}$ for which $R(A,n) \geqslant 1$ for all integers $n \geqslant 0$ and $R(A,n)$ is bounded in the square mean. Tang [10] gave a quantitative version of Ruzsa's theorem. Recently, the author and Yang [3] gave a new proof of Ruzsa's theorem.

In this Note, the following result is proved:

Theorem 1. *There is a basis A of $\mathbb{N}$ such that the set of n with $R(A,n)=2$ has density one.*

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2. Proofs

Lemma 1. *(See [7, Lemma 2].) Let $w_1, \ldots, w_s$ be s distinct nonnegative integers. If*

$$\sum_{i=1}^{s} 2^{w_i} = \sum_{j=1}^{t} 2^{x_j}$$

where $x_1, \ldots, x_t$ are nonnegative integers that are not necessarily distinct, then there is a partition of $\{1, 2, \ldots, t\}$ into s nonempty sets $J_1, \ldots, J_s$ such that

$$2^{w_i} = \sum_{j \in J_i} 2^{x_j}$$

for $i = 1, \ldots, s$.

Lemma 2. *Let w be a nonnegative integer, and let I and J be two finite sets of nonnegative integers such that the integers in $I \cup J$ have the same parity. If*

$$2^w = \sum_{i \in I} 2^i + \sum_{j \in J} 2^j, \tag{1}$$

then either $I \cup J = \{w\}$ or $I = J = \{w-1\}$.

Proof. If $I = \emptyset$ or $J = \emptyset$, then the conclusion is clear by the uniqueness of the binary representation. We now assume that $I \neq \emptyset$ and $J \neq \emptyset$. Let i_1 and j_1 be the least integers in I and J respectively. If $i_1 \neq j_1$, say $i_1 < j_1$, then by (1) we have

$$-2^{i_1} = \sum_{i \in I \setminus \{i_1\}} 2^i + \sum_{j \in J} 2^j - 2^w. \tag{2}$$

The right-hand side of (2) is divisible by 2^{i_1+1}, a contradiction. So $i_1 = j_1$. Thus

$$2^w - 2^{i_1+1} = \sum_{i \in I \setminus \{i_1\}} 2^i + \sum_{j \in J \setminus \{j_1\}} 2^j. \tag{3}$$

Suppose that $w > i_1 + 1$. Since the integers in $I \cup J$ have the same parity, the right-hand side of (3) is divisible by 2^{i_1+2}. But the left-hand side of (3) is not divisible by 2^{i_1+2}, a contradiction. Hence $w = i_1 + 1$. Thus $I = \{i_1\} = \{w-1\}$ and $J = \{j_1\} = \{w-1\}$. □

Let P be a possible property of a positive integer, and $P(x)$ the number of positive integers less than x with the property P. If $P(x)/x \to 1$ as $x \to \infty$, we say that *almost all positive integers possess the property P*.

Lemma 3. *(See [6, Theorem 143].) Almost all positive integers, when expressed in any scale, contain a given possible sequence of digits.*

Proof of Theorem 1. Let

$$A = \left\{ \sum_{i=0}^{\infty} \varepsilon_i 2^{2i} \colon \varepsilon_i \in \{0,1\} \right\} \cup \left\{ \sum_{i=1}^{\infty} \varepsilon_i 2^{2i-1} \colon \varepsilon_i \in \{0,1\} \right\},$$

where in each sum there are only finitely many $\varepsilon_i = 1$. Since each positive integer has its binary representation and $0 \in A$, it follows that $R(A, n) \geqslant 1$ for all integers $n \geqslant 0$. We say that a positive integer n has the property P if n contains a sequence 111 in its binary representation. By Lemma 3, almost all positive integers have the property P. In order to prove Theorem 1, it is enough to prove that $R(A, n) = 2$ for all n with the property P.

Let $n = \sum_{i \in I} 2^i$ be a positive integer with the property P. We treat the case where $\{2k, 2k+1, 2k+2\} \subseteq I$, for a certain $k \geqslant 0$. The case where $\{2k+1, 2k+2, 2k+3\} \subseteq I$, for a certain $k \geqslant 0$, can be treated similarly.

Let $n = a' + a''$ with $a', a'' \in A$. It is clear that $a' \neq 0$ and $a'' \neq 0$.

We suppose that

$$a' = \sum_{i \in I'} 2^{2i}, \qquad a'' = \sum_{i \in I''} 2^{2i}$$

and we shall obtain a contradiction.

By Lemma 1, there are two disjoint subsets I'_1, I'_2 of I' and two disjoint subsets I''_1, I''_2 of I'' (possibly $I'_j = \emptyset$ and $I''_j = \emptyset$, $j = 1$ or 2) such that

$$2^{2k} = \sum_{i \in I'_1} 2^{2i} + \sum_{i \in I''_1} 2^{2i}$$

and

$$2^{2k+1} = \sum_{i \in I'_2} 2^{2i} + \sum_{i \in I''_2} 2^{2i}.$$

By Lemma 2 we have $I'_1 \cup I''_1 = I'_2 = I''_2 = \{k\}$. This contradicts the fact that $I'_1 \cap I'_2 = \emptyset$ and $I''_1 \cap I''_2 = \emptyset$.

Similarly, we can derive a contradiction (using $2k+1$ and $2k+2$) if

$$a' = \sum_{i \in I'} 2^{2i+1}, \qquad a'' = \sum_{i \in I''} 2^{2i+1}.$$

By the uniqueness of the binary representation and the definition of A, we have that either

$$a' = \sum_{i \in I, 2|i} 2^i, \qquad a'' = \sum_{i \in I, 2\nmid i} 2^i$$

or

$$a'' = \sum_{i \in I, 2|i} 2^i, \qquad a' = \sum_{i \in I, 2\nmid i} 2^i.$$

Therefore, $R(A, n) = 2$. □

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