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Proof of the Kontsevich non-commutative cluster positivity conjecture

Démonstration de la conjecture de positivité de Kontsevich pour les graines non commutatives

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ABSTRACT

We extend the Lee–Schiffler Dyck path model to give a proof of the Kontsevich non-commutative cluster positivity conjecture with unequal parameters.

RÉSUMÉ

Nous étendons le modèle des chemins de Dyck, introduit par Lee–Schiffler, pour donner une preuve de la conjecture de positivité de Kontsevich pour les graines non commutatives à paramètres inégaux.

Let k be any field of characteristic zero. For any $r \in \mathbb{Z}_{>0}$, consider the following k-linear automorphism of the skew-field $K = k(x, y)$ of rational functions in non-commutative variables x and y:

$$F_r : (x, y) \mapsto \left(xyx^{-1}, \left(1 + y^r\right)x^{-1}\right).$$

This is a non-commutative analogue of the cluster mutations from [5]. In particular our main result establishes the positivity conjecture for rank two cluster algebras and quantum cluster algebras [2]. These automorphisms also fit into the framework of rank two non-commutative cluster algebras [4] and our main result completes the proof of the positive Laurent conjecture.

Theorem 1 *(Kontsevich conjecture). For any $r_1, r_2 \in \mathbb{Z}_{>0}$ and any $k \geqslant 0$, the elements $x_k = \underbrace{F_{r_1} F_{r_2} F_{r_1} \cdots}_{k}(x)$ are given by non-commutative Laurent polynomials in x and y with non-negative integer coefficients.*

Remark 2. Using a symmetry argument, Theorem 1 implies an analogous statement for $y_k = \underbrace{F_{r_1} F_{r_2} F_{r_1} \cdots}_{k}(y)$.

The Laurentness of these expressions was established by Usnich [7] for $r_1 = r_2$ and by Berenstein and Retakh [1] for general r_1, r_2. The positivity was shown by Di Francesco and Kedem [3] for $r_1 r_2 = 4$ and by Lee and Schiffler [6] for $r_1 = r_2$. We follow the Lee–Schiffler approach in this note.

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Fix integers $r_1, r_2 \in \mathbb{Z}_{>0}$. Our proof will make use of two-parameter Chebyshev polynomials $U_{k,j}$, $k, j \in \mathbb{Z}$, defined recursively by: $U_{-1,j} = 0$, $U_{0,j} = 1$, $U_{k+1,j+1} = r_j U_{k,j} - U_{k-1,j-1}$, where $r_j = \begin{cases} r_1, & \text{if } j \text{ is odd;} \\ r_2, & \text{if } j \text{ is even.} \end{cases}$ From now on we will work under the assumption $r_1 r_2 \geqslant 5$. The cases $r_1 r_2 \in \{1, 4\}$ were settled in [7] and [3] and the remaining cases $r_1 r_2 \in \{2, 3\}$ are given explicitly at http://pages.uoregon.edu/drupel/dyck_examples.pdf.

Fix $n \geqslant 2$. Consider the rectangle $R_n \subset \mathbb{Z}^2$ with corner vertices $(0,0)$ and $(U_{n-3,1} - U_{n-4,2}, U_{n-4,2})$. When R_n lies in the first quadrant, a *Dyck path* is a lattice path in R_n starting at $(0,0)$ and taking North or East steps to end at $(U_{n-3,1} - U_{n-4,2}, U_{n-4,2})$ such that the path never crosses the main diagonal of R_n and the slope of each subpath beginning at $(0,0)$ does not exceed the slope of the main diagonal. Here we consider a vertical edge to have slope ∞. We modify this definition slightly when R_n lies in the second quadrant by replacing the East step with a diagonal $(-1,1)$-upstep and considering vertical edges to have slope $-\infty$. When $n = 2$, R_n lies in the fourth quadrant and we use a diagonal $(1,-1)$-downstep. We will call a Dyck path *maximal* if no subpath of another Dyck path lies closer to the main diagonal. Write D_n for the maximal Dyck path in R_n. The next lemma follows by induction from the definitions:

Lemma 3. *Denote* $\epsilon_k := \max\{0, 2 - r_{k-1}\}$, $\delta_k := \epsilon_k + 2\epsilon_{k-1} + 1$ *for* $k \in \mathbb{Z}$. *Suppose* $k - \delta_k \geqslant 4$. *Then the Dyck path* D_k *consists of* $r_{k-\epsilon_{k-1}} - \delta_k + 1$ *copies of* $D_{k-1-\epsilon_{k-1}}$ *followed by a copy of* $D_{k-1-\epsilon_{k-1}}$ *with its first* $D_{k-1-\delta_k}$ *removed.*

Let $U_n = \max\{|U_{n-3,1}|, |U_{n-4,2}|\}$ be the number of edges in $D_n = (\omega_0, \alpha_1, \omega_1, \alpha_2, \ldots, \alpha_{U_n}, \omega_{U_n})$, where the vertices of D_n are labeled by $\omega_0, \omega_1, \ldots, \omega_{U_n}$ and α_i is the edge connecting ω_{i-1} and ω_i. Let $i_1, \ldots, i_{U_{n-4,2}}$ denote the increasing sequence so that α_{i_j} makes an upward step. We will write $\nu_0, \ldots, \nu_{U_{n-4,2}}$ for the sequence of vertices satisfying $\nu_0 = (0,0)$ and $\nu_j = \omega_{i_j}$.

Definition 4. For $i < j$ denote by s_{ij} the slope of the line from ν_i to ν_j and by s the slope of the main diagonal of R_n. For $0 \leqslant i < k \leqslant U_{n-4,2}$ let $\alpha(i,k)$ be the subpath of D_n from ν_i to ν_k labeled/colored as follows:

(1) If $s_{it} \leqslant s$ for all t with $i < t \leqslant k$, then $\alpha(i,k)$ is called a *Dyck prefix* (blue).
(2) If $s_{it} > s$ for some t with $i < t \leqslant k$, then
   (a) if the *smallest* such t is of the form $i + U_{m,2} - wU_{m-1-\epsilon_{m-1},2}$ for some integers $1 \leqslant m \leqslant n-4$ and $1 \leqslant w < r_{m-\epsilon_{m-1}} - \delta_m$, then $\alpha(i,k)$ is called an (m,w)-Dyck suffix (green).
   (b) otherwise, $\alpha(i,k)$ is called a *short suffix* (red).

Write $\mathcal{P}(D_n) = \{\alpha(i,k)\colon\ 0 \leqslant i < k \leqslant U_{n-4,2}\} \cup \{\alpha_1, \ldots, \alpha_{U_n}\}$ for the set of admissible subpaths of D_n. For $\beta \subset \mathcal{P}(D_n)$ we define the support $\mathrm{supp}(\beta) \subset D_n$ in the natural way. We will use the term *hook* for the supports of the subpaths $\alpha(k, k+1)$. It will be convenient to refer to a hook as type 1, 2, or 3 depending on whether the horizontal displacement from the bottom to the top of the hook is $r_2 - 1$, $r_2 - 2$, or $r_2 - 3$, respectively.

Call $\beta \subset \mathcal{P}(D_n)$ an *overlapping* collection if there exist either $\alpha(i,k), \alpha(i',k') \in \beta$ which share a vertex or $\alpha_j, \alpha(i,k) \in \beta$ with $\alpha_j \in \alpha(i,k)$. We will need the following K-valued weightings on non-overlapping collections:

Definition 5. Write

$$\varepsilon_i = \begin{cases} 1 & \text{if } \alpha_i \text{ is vertical;} \\ 0 & \text{otherwise.} \end{cases}$$

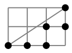
For each non-overlapping collection $\beta \subset \mathcal{P}(D_n)$ define

$$\beta_{[i]} = \begin{cases} y^{r_1 - \varepsilon_i} x^{-1}, & \text{if } \alpha_i \notin \mathrm{supp}(\beta); \\ y^{-\varepsilon_i} x^{-1}, & \text{if } \alpha_i \in \beta \text{ and } \alpha_i \text{ is not diagonal;} \\ x^1 y^{-1} x^{-1}, & \text{if } \alpha_i \in \beta \text{ and } \alpha_i \text{ is diagonal with an upstep;} \\ x^0 y^1, & \text{if } \alpha_i \in \beta \text{ and } \alpha_i \text{ is diagonal with a downstep;} \\ x^0 y^0, & \text{if } \alpha_i \in \alpha(j,k) \in \beta \text{ is horizontal;} \\ x^h y^{-1} x^{-1}, & \text{if } \alpha_i \in \alpha(j,k) \in \beta \text{ is the last edge of a hook of type h.} \end{cases}$$

We have the following refinement of Theorem 1:

Theorem 6. *Suppose* $r_1, r_2 \in \mathbb{Z}_{>0}$. *Write* $q = xyx^{-1}y^{-1}$. *Then for* $n \geqslant 2$ *we have* $x_{n-1} = \sum_{\beta \in \mathcal{F}(D_n)} q \prod_{i=1}^{U_n} \beta_{[i]}$, *where the product is taken in the natural order and the sum ranges over the set* $\mathcal{F}(D_n)$ *of non-overlapping collections* $\beta \subset \mathcal{P}(D_n)$ *subject to the conditions*:

$C1$: *if* α_i *is diagonal, then* α_i *is supported on* β;
$C2$: *if* $\alpha(i,k) \in \beta$ *is a short suffix, then the preceding non-diagonal edge of* ν_i *is supported on* β;
$C3$: *if* $\alpha(i,k) \in \beta$ *is an* (m,w)*-Dyck suffix, then at least one of the preceding* $U_{m-1,1} - wU_{m-2-\epsilon_{m-1},1}$ *non-diagonal edges of* ν_i *is supported on* β.

Example 7. For $r_1 = 2$, $r_2 = 3$, $n = 5$ we have $U_{2,1} = 5$, $U_{1,2} = 2$ and so R_5 and D_5 are given by: . We have the following expression for x_4:

$$x_4 = qxy^{-1}xy^{-1}x^{-1} + qxy^{-1}x^{-1}(1+y^2)x^{-1}(1+y^2)y^{-1}x^{-1} + q(1+y^2)x^{-1}(1+y^2)x^{-1}y^{-1}xy^{-1}x^{-1}$$
$$+ q(1+y^2)x^{-1}(1+y^2)x^{-1}(1+y^2)y^{-1}x^{-1}(1+y^2)x^{-1}(1+y^2)y^{-1}x^{-1},$$

where a factor of $1+y^2$ indicates an edge which may be either included in or excluded from the corresponding admissible collection of labeled/colored subpaths. We present several examples for $r_1r_2 = 5$, enumerating all admissible collections with their monomials, at http://pages.uoregon.edu/drupel/dyck_examples.pdf.

We divide the proof of Theorem 6 into a series of lemmas. First we make the following definitions:

Definition 8. Define the set $\tilde{\mathcal{F}}(D_n)$ of non-overlapping collections $\beta \subset \mathcal{P}(D_n)$ subject to conditions $C1$ and $C2$. Define $\mathcal{T}^{\geqslant u}(D_n) \subset \tilde{\mathcal{F}}(D_n)$ to consist of those β satisfying the following condition only for $m \geqslant u$:

$C3^{op}$: there exist integers i, k, w, m such that $\alpha(i,k) \in \beta$ is an (m,w)-Dyck suffix and none of the preceding $U_{m-1,1} - wU_{m-2-\epsilon_{m-1},1}$ non-diagonal edges of ν_i are supported on β.

Lemma 9. *If $m \geqslant n-3$, there do not exist i, w ($1 \leqslant w < r_{m-\epsilon_{m-1}} - \delta_m$) so that $\min\{t\colon\ i < t \leqslant U_{n-4,2},\ s_{i,t} > s\}$ is of the form $i + U_{m,2} - wU_{m-1-\epsilon_{m-1},2}$. In particular, for any $n \geqslant 2$, the set $\mathcal{T}^{\geqslant n-3}(D_n)$ is empty.*

Proof. We assume $\epsilon_{m-1} = 0$; the case $\epsilon_{m-1} > 0$ follows from this one. Since $w < r_m - 1 - \epsilon_m$, we have

$$U_{m,2} - wU_{m-1,2} \geqslant U_{m,2} - r_mU_{m-1,2} + (2+\epsilon_m)U_{m-1,2} = (2+\epsilon_m)U_{m-1,2} - U_{m-2,2} \geqslant U_{m-k,2}, \quad \text{for } k \geqslant 1.$$

Now if $m \geqslant n-3$ and $\tau := \min\{t\colon\ i < t \leqslant U_{n-4,2},\ s_{i,t} > s\} = i + U_{m,2} - wU_{m-1,2}$, then $\tau \geqslant i + U_{n-4,2}$. But this contradicts $\nu_{U_{n-4,2}}$ being the highest labeled vertex in D_n. □

Let $z_0 = x_0 = x$ and for $n \geqslant 2$ write $z_{n-1} = \sum_{\beta \in \tilde{\mathcal{F}}(D_n)} q \prod_{i=1}^{U_n} \beta_{[i]}$. For each integer ℓ we will use a parenthesized exponent $^{(\ell)}$ to denote a quantity with each r_k replaced by $r_{k+\ell}$. In particular, note that $F_{r_2}(x_k) = x_{k+1}^{(1)}$.

Lemma 10. *Suppose $n \geqslant 2$. Then $z_n^{(1)} = F_{r_2}(z_{n-1}) + \sum_{\beta \in \mathcal{T}^{\geqslant 1}(D_{n+1}^{(1)}) \setminus \mathcal{T}^{\geqslant 2}(D_{n+1}^{(1)})} q \prod_{i=1}^{U_{n+1}^{(1)}} \beta_{[i]}$.*

Proof. This follows from a study of how the $(1+y^{r_2})^{-1}$ terms cancel in $F_{r_2}(z_{n-1})$. In particular, we make the following observations. The sum of the weights of a colored hook and the corresponding full hook of uncolored edges gives rise to a Laurent monomial under F_{r_2}. An edge α in the support of β gives rise to a colored hook of type 1, 2, or 3 corresponding to the edge α being horizontal, vertical not followed by a diagonal, or vertical followed by a diagonal, respectively. A missing edge α gives rise to all collections of uncolored edges in a hook of type 1, 2, or 3 corresponding to the edge α being horizontal, vertical not followed by a diagonal, or vertical followed by a diagonal, respectively.

Now consider an uncolored hook with a missing horizontal edge, followed by d included horizontal edges, and then an included vertical edge. Under F_{r_2} the weight of this configuration gives rise to the weights of all collections of horizontal edges in a hook of type 1 with an included vertical edge followed by d colored hooks of type 1 and then a colored hook of type 2. The sum is accounting for the included vertical edge in this case. □

In the following lemma we consider a D_3 with its first D_2 removed as a single vertical edge and for $\epsilon_3 = 1$ we consider a D_4 with its first D_2 removed as a vertical edge followed by a $(-1,1)$-diagonal edge.

Lemma 11.

(1) *Suppose $k - \epsilon_{k-1} \geqslant 5$. Then the weight of a missing D_{k-2} with its first $D_{k-3-\epsilon_{k-3}}$ removed followed by a colored D_k simplifies to the weight of a colored $D_{k-1-\epsilon_{k-1}}$.*
(2) *Suppose $k - \epsilon_{k-1} \geqslant 5$. Then the weight of a missing D_{k-2} followed by a colored $D_{k-1-\epsilon_{k-1}}$ simplifies to the weight of a missing D_{k-2} with its first $D_{k-3-\epsilon_{k-3}}$ removed.*
(3) *Suppose $m - \delta_m \geqslant 0$. Then for $1 \leqslant w < r_{m-\epsilon_{m-1}} - \delta_m$, the weight of an (m,w)-Dyck suffix preceded by $U_{m-1,1} - wU_{m-2-\epsilon_{m-1},1}$ missing non-diagonal edges is equal to the weight of an $(m,w+1)$-Dyck suffix preceded by $U_{m-1,1} - (w+1)U_{m-2-\epsilon_{m-1},1}$ missing non-diagonal edges.*

Proof. Parts (1) and (2) follow from a simultaneous induction using Lemma 3 in the induction step. Part (3) follows from (1), (2), and Lemma 3. □

Corollary 12. *Suppose $m-\delta_m \geqslant 0$. Then for $1 \leqslant w < r_{m-\epsilon_{m-1}} - \delta_m$, the weight of an (m,w)-Dyck suffix preceded by $U_{m-1,1} - wU_{m-2-\epsilon_{m-1},1}$ missing non-diagonal edges is equal to q^{-1}.*

Proof. We work by induction, the case $m-\delta_m=0$ is easy to check by hand. It follows from Lemma 3 that the hook sequences of an $(m, r_{m-\epsilon_{m-1}} - \delta_m)$-Dyck suffix and an $(m-1,1)$-Dyck suffix are the same. Then one easily checks that $U_{m-1,1} - (r_{m-\epsilon_{m-1}} - \delta_m)U_{m-2-\epsilon_{m-1},1} = U_{m-2,1} - U_{m-3-\epsilon_{m-2},1}$, the case $\epsilon_{m-1}>0$ following from the case $\epsilon_{m-1}=0$. The result now follows by induction using Lemma 11(3). □

Lemma 13. *Let $u \geqslant 1$ and $n \geqslant u+4$. Then*

$$F_{r_2}\left(\sum_{\beta\in\mathcal{T}^{\geqslant u}(D_n)\setminus\mathcal{T}^{\geqslant u+1}(D_n)} q\prod_{i=1}^{U_n}\beta_{[i]}\right) = \sum_{\beta\in\mathcal{T}^{\geqslant u+1}(D_{n+1}^{(1)})\setminus\mathcal{T}^{\geqslant u+2}(D_{n+1}^{(1)})} q\prod_{i=1}^{U_{n+1}^{(1)}}\beta_{[i]}.$$

Proof. The proof follows by simultaneous induction with Lemma 14. We will assume $n=u+4$, the case $n>u+4$ follows from this one using a similar argument. Also we restrict to the case $\epsilon_{n-1}=0$, the case $\epsilon_{n-1}>0$ follows by a similar argument.

From Lemma 3, we can see that D_n begins with w copies of D_{n-1}, $1 \leqslant w < r_n - 1 - \epsilon_n$, and the vertex $\nu_{wU_{n-5,2}}$ is the ending vertex of the last D_{n-1}. Now $\alpha(wU_{n-5,2}, U_{n-4,2})$ is the only $(n-4,w)$-Dyck suffix of D_n and so $\beta\in\mathcal{T}^{\geqslant n-4}(D_n)$ implies $\alpha(wU_{n-5,2}, U_{n-4,2})\in\beta$ and none of the preceding $U_{n-5,1} - wU_{n-6,1}$ non-diagonal edges are contained in β. Note that $wU_{n-4,1} - U_{n-5,1} + wU_{n-6,1} = r_2 wU_{n-5,2} - U_{n-5,1}$ and so the lowest vertex of these missing edges is $\omega_{r_2 wU_{n-5,2}-U_{n-5,1}}$. Then Lemma 3 implies the subpath of D_n from ω_0 to $\omega_{r_2 wU_{n-5,2}-U_{n-5,1}}$ consists of $w-1$ copies of D_{n-1}, followed by $r_{n-1}-1$ copies of D_{n-2}, and then $w-1$ copies of D_{n-3}. We will define j_i for $0\leqslant i\leqslant 2w+r_{n-1}-3$ so that the ν_{j_i} are the endpoints of these copies. Any subpath $\alpha(i,k)$ can be decomposed as $\alpha(i,j_e), \alpha(j_e,j_{e+1}),\ldots,\alpha(j_{e+\ell},k)$ where all but the first are Dyck prefixes. It is easy to see that $\alpha(i,j_e)$ has the same label/color as $\alpha(i,k)$ and if $\alpha(i,k)$ was an (m,w')-Dyck suffix then so is $\alpha(i,j_e)$.

Combining the above considerations we see that $\sum_{\beta\in\mathcal{T}^{\geqslant u}(D_n)\setminus\mathcal{T}^{\geqslant u+1}(D_n)} q\prod_{i=1}^{U_n}\beta_{[i]}$ can be rewritten as:

$$\sum_{w=1}^{r_n-2-\epsilon_n} q\left(\sum_{\beta\in\mathcal{F}(D_{n-1})}\prod_{i=1}^{U_{n-1}}\beta_{[i]}\right)^{w-1}\left(\sum_{\beta\in\mathcal{F}(D_{n-2})}\prod_{i=1}^{U_{n-2}}\beta_{[i]}\right)^{r_{n-1}-1}\left(\sum_{\beta\in\mathcal{F}(D_{n-3})}\prod_{i=1}^{U_{n-3}}\beta_{[i]}\right)^{w-1} q^{-1}$$
$$= \sum_{w=1}^{r_n-2-\epsilon_n} q\left(q^{-1}x_{n-2}\right)^{w-1}\left(q^{-1}x_{n-3}\right)^{r_{n-1}-1}\left(q^{-1}x_{n-4}\right)^{w-1}q^{-1},$$

where the equality follows from Lemma 14. Applying F_{r_2} and noting that $F_{r_2}(q)=q$ completes the proof. □

Lemma 14. *Suppose $n\geqslant 3$. Then*

$$x_{n-1} = z_{n-1} - \sum_{m=5}^{n}\underbrace{F_{r_1}F_{r_2}F_{r_1}\cdots}_{n-m}\left(\sum_{\beta\in\mathcal{T}^{\geqslant 1}(D_m^{(m-n)})\setminus\mathcal{T}^{\geqslant 2}(D_m^{(m-n)})} q\prod_{i=1}^{U_m^{(m-n)}}\beta_{[i]}\right) = \sum_{\beta\in\mathcal{F}(D_n)} q\prod_{i=1}^{U_n}\beta_{[i]}. \quad (1)$$

Proof. This follows from simultaneous induction with Lemma 13 as in the proof of [6, Lemma 20]. □

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References

[1] A. Berenstein, V. Retakh, A short proof of Kontsevich cluster conjecture, C. R. Acad. Sci. Paris, Ser. I 349 (3–4) (2011) 119–122.
[2] A. Berenstein, A. Zelevinsky, Quantum cluster algebras, Adv. Math. 195 (2) (2005) 405–455.
[3] P. Di Francesco, R. Kedem, Discrete non-commutative integrability: Proof of a conjecture of M. Kontsevich, Int. Math. Res. Not. 2010 (21) (2010) 4042–4063, http://dx.doi.org/10.1093/imrn/rnq024.
[4] P. Di Francesco, R. Kedem, Non-commutative integrability, paths and quasi-determinants, Adv. Math. 228 (1) (2011) 97–152.
[5] S. Fomin, A. Zelevinsky, Cluster algebras I: Foundations, J. Amer. Math. Soc. 15 (2) (2002) 497–529.
[6] K. Lee, R. Schiffler, Proof of a positivity conjecture of M. Kontsevich on non-commutative cluster variables, preprint, arXiv:1109.5130 [math.QA], 2011.
[7] A. Usnich, Non-commutative Laurent phenomenon for two variables, preprint, arXiv:1006.1211 [math.AG], 2010.