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Partial Differential Equations

Spectral instability of some non-selfadjoint anharmonic oscillators

Instabilité spectrale de certains oscillateurs anharmoniques non-autoadjoints

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ABSTRACT

The purpose of this Note is to highlight the spectral instability of some non-selfadjoint differential operators, by studying the growth rate of the norms of the spectral projections Π_n associated with their eigenvalues. More precisely, we are concerned with some *anharmonic oscillators* $\mathcal{A}(m,\theta) = -\frac{d^2}{dx^2} + e^{i\theta}|x|^m$ with $|\theta| < \min\{\frac{(m+2)\pi}{4}, \frac{(m+2)\pi}{2m}\}$, defined on $L^2(\mathbb{R})$. We get asymptotic expansions for the norm of the spectral projections associated with the large eigenvalues of $\mathcal{A}(1,\theta)$ and $\mathcal{A}(2k,\theta)$, $k \geqslant 1$, extending the results of Davies (2000) [4] and Davies and Kuijlaars (2004) [5].

RÉSUMÉ

Notre objectif est de mettre en évidence l'instabilité spectrale de certains opérateurs différentiels non-autoadjoints, via l'étude de la croissance des normes des projecteurs spectraux Π_n associés à leurs valeurs propres. Nous nous intéressons à certains *oscillateurs anharmoniques* $\mathcal{A}(m,\theta) = -\frac{d^2}{dx^2} + e^{i\theta}|x|^m$ avec $|\theta| < \min\{\frac{(m+2)\pi}{4}, \frac{(m+2)\pi}{2m}\}$, définis sur $L^2(\mathbb{R})$. Nous étendons les résultats de Davies (2000) [4] et Davies et Kuijlaars (2004) [5] en donnant un développement asymptotique de la norme des projecteurs spectraux associés aux grandes valeurs propres pour les opérateurs $\mathcal{A}(1,\theta)$ et $\mathcal{A}(2k,\theta)$, $k \geqslant 1$.

1. Spectral instability and pseudospectra

It is well known that the spectral theorem implies some control of stability for the spectrum of selfadjoint operators: if $\mathcal{A}$ is a selfadjoint operator acting on the Hilbert space $\mathcal{H}$, the spectrum of its perturbations $\mathcal{A} + \varepsilon\mathcal{B}$, with $\varepsilon > 0$ and any $\mathcal{B} \in \mathcal{L}(\mathcal{H})$, $\|\mathcal{B}\| \leqslant 1$, lies entirely inside an ε-neighborhood of the spectrum $\sigma(\mathcal{A})$. In other words, the norm of the resolvent of $\mathcal{A}$ near the spectrum blows up like the inverse distance to the spectrum. It has also been known for several years (see for instance [13]) that such a behavior could not be expected in general in the case of non-selfadjoint operators. One can understand it thanks to the notion of ε-*pseudospectra* of an operator $\mathcal{A}$, defined as the family of sets $\sigma_\varepsilon(\mathcal{A})$, indexed by $\varepsilon > 0$,

$$\sigma_\varepsilon(\mathcal{A}) = \left\{\xi \in \rho(\mathcal{A}):\ \left\|(\mathcal{A} - \xi)^{-1}\right\| > \frac{1}{\varepsilon}\right\} \cup \sigma(\mathcal{A}).$$

The link between spectral instability and pseudospectra appears more clearly in the following equivalent formulation, which is a weak version of the Roch and Silbermann theorem [11]:

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$$\sigma_\varepsilon(\mathcal{A}) = \bigcup_{\substack{\omega\in\mathcal{L}(\mathcal{H}),\\ \|\mathcal{B}\|\leqslant 1}} \sigma(\mathcal{A}+\varepsilon\mathcal{B})$$

(see also [12] and references therein).

In the following, we deal with the instability indices associated with an isolated eigenvalue $\lambda \in \sigma(\mathcal{A})$. The instability index associated with λ is defined as $\kappa(\lambda) = \|\Pi(\lambda)\|$, where $\Pi(\lambda)$ denotes the spectral projection associated with λ. Of course $\kappa(\lambda) \geqslant 1$ in any case, and $\kappa(\lambda) = 1$ when $\mathcal{A}$ is selfadjoint. These numbers $\kappa(\lambda)$ are closely related to the size of ε-pseudospectra around λ. Indeed, if $\sigma_\varepsilon^\lambda$ denotes the connected component of $\sigma_\varepsilon(\mathcal{A})$ containing λ, and if we assume for simplicity that $\sigma_\varepsilon^\lambda \cap \sigma(\mathcal{A}) = \{\lambda\}$ and $\sigma_\varepsilon^\lambda$ is bounded, then the perimeter $|\partial\sigma_\varepsilon^\lambda|$ of $\sigma_\varepsilon^\lambda$ satisfies (see [3])

$$\left|\partial\sigma_\varepsilon^\lambda\right| \geqslant 2\pi\varepsilon\kappa(\lambda). \tag{1}$$

In the finite dimensional setting at least, instability indices give a better description of pseudospectra: if $\mathcal{A} \in \mathcal{M}_n(\mathbb{C})$ is a diagonalizable matrix with distinct eigenvalues $\lambda_1, \dots, \lambda_n$, Embree and Trefethen show [13] that there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon < \varepsilon_0$,

$$\bigcup_{\lambda_k\in\sigma(\mathcal{A})} D\big(\lambda_k, \varepsilon\kappa(\lambda_k) + \mathcal{O}(\varepsilon^2)\big) \subset \sigma_\varepsilon(\mathcal{A}) \subset \bigcup_{\lambda_k\in\sigma(\mathcal{A})} D\big(\lambda_k, \varepsilon\kappa(\lambda_k) + \mathcal{O}(\varepsilon^2)\big). \tag{2}$$

In the case of an infinite dimensional space, the validity of this statement should be investigated, as well as the dependance on λ_k of the $\mathcal{O}(\varepsilon^2)$ terms.

In the following, we study the instability indices of simple non-selfadjoint differential operators introduced by Davies in [4], for which the instability phenomenon described above will appear clearly. Let us define the *anharmonic oscillators*

$$\mathcal{A}(m,\theta) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + e^{i\theta}|x|^m \tag{3}$$

with

$$|\theta| < \min\left\{\frac{(m+2)\pi}{4}, \frac{(m+2)\pi}{2m}\right\}, \tag{4}$$

defined on $L^2(\mathbb{R})$ in [4] by taking the closure of the associated quadratic form defined on $\mathcal{C}_0^\infty(\mathbb{R})$, which is sectorial if θ satisfies (4). According to [4], its spectrum consists of a sequence of discrete simple eigenvalues, denoted in nondecreasing modulus order by $\lambda_n = \lambda_n(m,\theta)$, $|\lambda_n| \to +\infty$. The associated spectral projections are of rank 1, and E.-B. Davies showed in [4] that for all $m \in]0, +\infty[$ and $\theta \neq 0$ satisfying (4), for all $\alpha > 0$, there exists $N = N(m,\theta,\alpha) \geqslant 0$ such that the instability indices $\kappa_n(m,\theta)$ of $\mathcal{A}(m,\theta)$ satisfy $\kappa_n(m,\theta) \geqslant n^\alpha$ for $n \geqslant N$. This statement has been improved in the case $m = 2$ of the harmonic oscillator (sometimes referred as the Davies operator), since E.-B. Davies and A. Kuijlaars showed [5] that $\kappa_n(2,\theta)$ grows exponentially fast as $n \to +\infty$, with an explicit rate $c(\theta)$: there exists an explicit $c(\theta)$ such that

$$\lim_{n\to+\infty} \frac{1}{n}\log\kappa_n(2,\theta) = c(\theta). \tag{5}$$

The purpose of this Note is to prove that this statement actually holds for the so-called *complex Airy operator* $\mathcal{A}(1,\theta)$ and for the even anharmonic oscillators $\mathcal{A}(2k,\theta)$, $k \geqslant 1$.

2. Non-selfadjoint anharmonic oscillators

We first deal with the complex Airy operator $\mathcal{A}(1,\theta)$ defined in (3). We show that the corresponding instability indices $\kappa_n(1,\theta)$ grow like in (5) as $n \to +\infty$. More precisely, we get asymptotic expansions in powers of n^{-1} as $n \to +\infty$.

Theorem 2.1. *Let $0 < |\theta| < 3\pi/4$. There exists a real sequence $(\alpha_j(\theta))_{j\geqslant 1}$ such that the instability indices $\kappa_n(1,\theta)$ of $\mathcal{A}(1,\theta)$ satisfy, as $n \to +\infty$,*

$$\exp\big(-C(\theta)(n-1/2)\big)\kappa_n(1,\theta) = \frac{K(\theta)}{\sqrt{n}}\left(1 + \sum_{j=1}^{+\infty}\alpha_j(\theta)n^{-j}\right) + \mathcal{O}\big(n^{-\infty}\big), \tag{6}$$

where

$$C(\theta) = \pi m_\theta^{3/2}|\sin\theta| \quad \textit{and} \quad K(\theta) = \frac{1}{2\sqrt{3|\sin\theta|}m_\theta^{1/4}},$$

with

$$m_\theta = \sqrt{1 + \frac{\sin^2(2\theta/3)}{\sin^2\theta} - 2\frac{\cos(\theta/3)\sin(2\theta/3)}{\sin\theta}} > 0.$$

Sketch of the proof. Let us first recall that all the eigenvalues of $\mathcal{A}(m,\theta)$, $m \in \mathbb{N}$, have associated spectral projections of rank 1, see Lemma 5 in [4]. Hence, one can easily check that [3]

$$\kappa_n(m,\theta) = \frac{\|u_n\|^2}{\langle u_n, \bar{u}_n\rangle}, \tag{7}$$

where u_n denotes an eigenfunction associated with the n-th eigenvalue of $\mathcal{A}(m,\theta)$.

We get rid of the singularity of the potential at $x=0$ by decomposing $\mathcal{A}(1,\theta)$ into its Dirichlet and Neumann realizations $\mathcal{A}^D(1,\theta)$ and $\mathcal{A}^N(1,\theta)$ on $\mathbb{R}^+$. We then compute their instability indices

$$\kappa_n^{D/N}(1,\theta) = \frac{\int_{\mathbb{R}^+} |Ai(\mu_n^{D/N} + e^{i\theta/3}x)|^2\,\mathrm{d}x}{|\int_{\mathbb{R}^+} Ai(\mu_n^{D/N} + e^{i\theta/3}x)^2\,\mathrm{d}x|}, \tag{8}$$

given by formula (7), where $x \mapsto Ai(\mu_n^{D/N} + e^{i\theta/3}x)$ is the n-th eigenfunction of $\mathcal{A}^{D/N}(1,\theta)$, μ_n^D (resp. μ_n^N) being the n-th (negative) zero (resp. critical point) of the Airy function Ai (see [2,8]). We estimate the numerator in (8) by using the well-known asymptotic expansion of Ai at infinity in the complex plane (see [1]), and the Laplace method brings an $\exp(c_\theta|\mu_n^{D/N}|^{3/2})$ term in $\kappa_n^{D/N}(1,\theta)$, $c_\theta > 0$. The integral in the denominator of (8), after deformation of the path of integration by homotopy, is equal to

$$\int_{\mu_n^{D/N}}^{+\infty} Ai^2(x)\,\mathrm{d}x = Ai'^2\big(\mu_n^{D/N}\big) \tag{9}$$

(it is indeed straightforward, using Airy equation, to check that $x \mapsto xAi^2(x) - Ai'^2(x)$ is a primitive for Ai^2). Hence the expansion of $Ai'(-z)$ as $z \to +\infty$, given in [1], provides an asymptotic expansion for (9) in powers of $|\mu_n^{D/N}|^{-3/2}$. The statement follows from the behavior of $\mu_n^{D/N}$ as $n \to +\infty$, since we have asymptotic expansions for $(n-1/4)^{-2/3}|\mu_n^D|$ (resp. $(n-3/4)^{-2/3}|\mu_n^N|$) in powers of $(n-1/4)^{-2}$ (resp. $(n-3/4)^{-2}$). □

Notice that the exponential instability appears as soon as $\theta \neq 0$.

We have a similar statement for even anharmonic oscillators:

Theorem 2.2. *Let $k \in \mathbb{N}^*$ and θ be such that $0 < |\theta| < \frac{(k+1)\pi}{2k}$. If $\kappa_n(2k,\theta)$ denotes the n-th instability index of $\mathcal{A}(2k,\theta) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + e^{i\theta}x^{2k}$, then there exist $K(2k,\theta) > 0$ and a real sequence $(C^j(2k,\theta))_{j\geqslant 1}$ such that*

$$e^{-c_k(\theta)n}\kappa_n(2k,\theta) = \frac{K(2k,\theta)}{\sqrt{n}}\left(1 + \sum_{j=1}^{+\infty} C^j(2k,\theta)n^{-j}\right) + \mathcal{O}\big(n^{-\infty}\big) \tag{10}$$

as $n \to +\infty$, with

$$c_k(\theta) = \frac{2(k+1)\sqrt{\pi}\,\Gamma(\frac{k+1}{2k})\varphi_k(\xi_k)}{\Gamma(\frac{1}{2k})} > 0, \tag{11}$$

where

$$\xi_k = \left(\frac{\tan(|\theta|/(k+1))}{\sin(k|\theta|/(k+1)) + \cos(k|\theta|/(k+1))\tan(|\theta|/(k+1))}\right)^{\frac{1}{2k}}, \tag{12}$$

$$\varphi_k(\xi) = \operatorname{Im} \int_0^{\xi e^{i\frac{\theta}{2(k+1)}}} \big(1 - t^{2k}\big)^{1/2}\,\mathrm{d}t. \tag{13}$$

Sketch of the proof. We first perform an analytic dilation and a scale change to recover the semiclassical selfadjoint anharmonic oscillator $\mathcal{P}_h(2k) = -h^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^{2k} - 1$, with $h = h_n = |\lambda_n(2k,\theta)|^{-\frac{k+1}{2k}}$. The n-th instability index of $\mathcal{A}(2k,\theta)$ then writes

$$\kappa_n(2k,\theta) = \frac{\int_{\mathbb{R}} |\psi_h(e^{i\frac{\theta}{2(k+1)}}x)|^2\,\mathrm{d}x}{\int_{\mathbb{R}} \psi_h^2(x)\,\mathrm{d}x} \tag{14}$$

where ψ_h solves $\mathcal{P}_h(2k)\psi_h = 0$, $\psi_h \in L^2(\mathbb{R})$ (see (7), after deformation of the integration path in the denominator). The complex WKB method (see [10,14,6]) and the analysis of the Stokes lines of $\mathcal{P}_h(2k)$ provide an asymptotic expansion of

$\psi_h(e^{i\frac{\theta}{2(k+1)}}x)$ as $h \to 0$, which enables us to determine the asymptotic behaviour of the numerator in (14), using again the Laplace method.

On the real axis, ψ_h is treated separately in its oscillatory region $[-1+\delta, 1-\delta]$, $\delta > 0$, and in the neighbourhood of the turning points ± 1. Hence, the stationary phase method leads to an asymptotic expansion in powers of h of the denominator in (14). Finally, the statement follows from the Bohr–Sommerfeld quantization rule for h_n (see [7, Exercise 12.3]) or Weyl formula [9]. □

In the harmonic case $k = 1$ (Davies operator), the first term in (10) yields the Davies–Kuijlaars theorem [5]:

$$\lim_{n\to+\infty} \frac{1}{n} \log \|\Pi_n\| = c_1(\theta) = 4\varphi_1\left(\frac{1}{\sqrt{2\cos(\theta/2)}}\right) = 2\,\mathrm{Re}\, f\left(\frac{e^{i\theta/4}}{\sqrt{2\cos(\theta/2)}}\right)$$

where $f(z) = \log(z + \sqrt{z^2-1}) - z\sqrt{z^2-1}$.

3. Eigenfunctions and semigroups

The following theorem has been proved in [2] in the case of complex Airy operator $\mathcal{A}(1,\theta)$, and in [4] in the harmonic case ($k = 1$), as well as for $\mathcal{A}(2k,\theta)$, $k \geqslant 2$, $|\theta| < \frac{\pi}{2}$. The proof actually extends to any operator $\mathcal{A}(2k,\theta)$ with $|\theta| < \frac{(k+1)\pi}{2k}$:

Theorem 3.1. *For any $m = 1, 2k$, $k \geqslant 1$, and any θ satisfying (4), the eigenfunctions of $\mathcal{A}(m,\theta)$ form a complete set of the space $L^2(\mathbb{R})$.*

Notice however that the eigenfunctions of $\mathcal{A}(1,\theta)$ and $\mathcal{A}(2k,\theta)$, $k \geqslant 1$, cannot form a Riesz basis because of the growth of the instability indices as $n \to +\infty$.

Theorem 3.1 and the previous estimates enable us to study the convergence of the operator series defining the semigroup $e^{-t\mathcal{A}(m,\theta)}$ associated with $\mathcal{A}(m,\theta)$ when decomposed along the projections Π_n.

The following statement extends the result of [5] in the harmonic case.

Corollary 3.2. *Let $|\theta| \leqslant \pi/2$ and $e^{-t\mathcal{A}(m,\theta)}$ be the semigroup generated by $\mathcal{A}(m,\theta)$, $\lambda_n = \lambda_n(m,\theta)$ the eigenvalues of $\mathcal{A}(m,\theta)$, and $\Pi_n = \Pi_n(m,\theta)$ the associated spectral projections.*

Let $T(\theta) = c_1(\theta)/\cos(\theta/2)$, where $c_1(\theta)$ is the constant in (11). The series $\Sigma_{m,\theta}(t) = \sum_{n=1}^{+\infty} e^{-t\lambda_n(m,\theta)}\Pi_n(m,\theta)$ is not normally convergent in cases $m = 1$ for any $t > 0$, and $m = 2$ for $t < T(\theta)$; in cases $m = 2$ for $t > T(\theta)$, and $m = 2k$ for any $t > 0$, $k \geqslant 2$, the series converges normally towards $e^{-t\mathcal{A}(m,\theta)}$ and, for N sufficiently large and for some constants $C_1 = C_1(k,\theta)$ and $C_2 = C_2(\theta)$, the following estimate holds

$$\left\|e^{-t\mathcal{A}(m,\theta)}(I - \Pi_{<N})\right\| \leqslant \begin{cases} \frac{C_1}{\sqrt{N}} e^{c_k(\theta)n} \exp(-t\,\mathrm{Re}\,\lambda_N), & k \geqslant 2, \\ \frac{C_2}{\sqrt{N}} \exp(-2\cos(\theta/2)(t - T(\theta))N), & k = 1,\ t > T \end{cases} \tag{15}$$

where $\Pi_{<N} = \Pi_1 + \cdots + \Pi_{N-1}$ denote the projection on the first $N-1$ eigenspaces.

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