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Complex Analysis

Some preserving sandwich results of certain integral operators on multivalent meromorphic functions

Quelques résultats de conservation de la subordination pour certains opérateurs intégraux sur les fonctions méromorphes multi-valuées

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A B S T R A C T

In this paper, we obtain some subordination, superordination and sandwich-preserving results of a certain integral operator on p-valent meromorphic functions.

R É S U M É

Nous présentons des résultats de sub- et super-ordination simultanées pour certains opérateurs sur les fonctions méromorphes p-valuées.

1. Introduction

Let $H(\mathbb{U})$ be the class of functions analytic in $\mathbb{U}=\{z\in\mathbb{C}\colon\ |z|<1\}$ and $H[a,n]$ be the subclass of $H(\mathbb{U})$ consisting of functions of the form:

$$f(z)=a+a_nz^n+a_{n+1}z^{n+1}+\cdots,$$

with $H_0=H[0,1]$ and $H=H[1,1]$. Let $\sum_p$ denote the class of all p-valent meromorphic functions of the form:

$$f(z)=\frac{1}{z^p}+\sum_{k=1-p}^{\infty}a_kz^k\quad\bigl(p\in\mathbb{N}=\{1,2,3,\ldots\};\ z\in\mathbb{U}^*=\mathbb{U}\backslash\{0\}\bigr).\tag{1.1}$$

Let f and F be members of $H(\mathbb{U})$. The function $f(z)$ is said to be subordinate to $F(z)$, or $F(z)$ is said to be superordinate to $f(z)$, if there exists a function $\omega(z)$ analytic in $\mathbb{U}$ with $\omega(0)=0$ and $|\omega(z)|<1 (z\in\mathbb{U})$, such that $f(z)=F(\omega(z))$. In such a case, we write $f(z)\prec F(z)$. If F is univalent, then $f(z)\prec F(z)$ if and only if $f(0)=F(0)$ and $f(\mathbb{U})\subset F(\mathbb{U})$ (see [4,5]).

Let $\phi:\mathbb{C}^2\times\mathbb{U}\to\mathbb{C}$ and $h(z)$ be univalent in $\mathbb{U}$. If $p(z)$ is analytic in $\mathbb{U}$ and satisfies the first-order differential subordination:

$$\phi\bigl(p(z),zp'(z);z\bigr)\prec h(z),\tag{1.2}$$

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then $p(z)$ is a solution of the differential subordination (1.2). The univalent function $q(z)$ is called a dominant of the solutions of the differential subordination (1.2) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.2). A univalent dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants of (1.2) is called the best dominant. If $p(z)$ and $\phi(p(z), zp'(z); z)$ are univalent in $\mathbb{U}$ and if $p(z)$ satisfies first-order differential superordination:

$$h(z) \prec \phi\big(p(z), zp'(z); z\big), \tag{1.3}$$

then $p(z)$ is a solution of the differential superordination (1.3). An analytic function $q(z)$ is called a subordinant of the solutions of the differential superordination (1.3) if $q(z) \prec p(z)$ for all $p(z)$ satisfying (1.3). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants of (1.3) is called the best subordinant (see [4,5]).

For a function f in the class $\sum_p$ given by (1.1), Aqlan et al. [1] introduced the following one-parameter family of integral operators:

$$\mathcal{P}_p^\alpha f(z) = \frac{1}{z^{p+1}\Gamma(\alpha)} \int_0^z \left(\log \frac{z}{t}\right)^{\alpha-1} t^{\alpha-1} f(t)\,\mathrm{d}t \quad (\alpha > 0;\ p \in \mathbb{N}). \tag{1.4}$$

Using an elementary integral calculus, it is easy to verify that (see [1]):

$$\mathcal{P}_p^\alpha f(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} \left(\frac{1}{k+p+1}\right)^{\alpha} a_k z^k \quad (\alpha \geqslant 0;\ p \in \mathbb{N}). \tag{1.5}$$

Also, it is easily verified from (1.5) that

$$z\big(\mathcal{P}_p^\alpha f(z)\big)' = \mathcal{P}_p^{\alpha-1} f(z) - (1+p)\mathcal{P}_p^\alpha f(z). \tag{1.6}$$

To prove our results, we need the following definitions and lemmas.

Definition 1. (See [4].) Denote by $\mathcal{F}$ the set of all functions $q(z)$ that are analytic and injective on $\bar{\mathbb{U}} \backslash E(q)$ where

$$E(q) = \Big\{\zeta \in \partial\mathbb{U}\text{: } \lim_{z \to \zeta} q(z) = \infty\Big\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U} \backslash E(q)$. Further let the subclass of $\mathcal{F}$ for which $q(0) = a$ be denoted by $\mathcal{F}(a)$, $\mathcal{F}(0) \equiv \mathcal{F}_0$ and $\mathcal{F}(1) \equiv \mathcal{F}$.

Definition 2. (See [5].) A function $L(z,t)$ $(z \in \mathbb{U}, t \geqslant 0)$ is said to be a subordination chain if $L(0,t)$ is analytic and univalent in $\mathbb{U}$ for all $t \geqslant 0$, $L(z,0)$ is continuously differentiable on $[0;1)$ for all $z \in \mathbb{U}$ and $L(z,t_1) \prec L(z,t_2)$ for all $0 \leqslant t_1 \leqslant t_2$.

Lemma 1. *(See [6].) The function $L(z,t) : \mathbb{U} \times [0;1) \longrightarrow \mathbb{C}$, of the form:*

$$L(z,t) = a_1(t)z + a_2(t)z^2 + \cdots \quad \big(a_1(t) \neq 0;\ t \geqslant 0\big),$$

and $\lim_{t\to\infty} |a_1(t)| = \infty$ is a subordination chain if and only if

$$\operatorname{Re}\left\{\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right\} > 0 \quad (z \in \mathbb{U}, t \geqslant 0).$$

Lemma 2. *(See [2].) Suppose that the function $H : \mathbb{C}^2 \to \mathbb{C}$ satisfies the condition:*

$$\operatorname{Re}\big\{H(is;t)\big\} \leqslant 0$$

for all real s and for all $t \leqslant -n(1+s^2)/2$, $n \in \mathbb{N}$. If the function $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \cdots$ is analytic in $\mathbb{U}$ and

$$\operatorname{Re}\big\{H\big(p(z); zp'(z)\big)\big\} > 0 \quad (z \in \mathbb{U}),$$

then $\operatorname{Re}\{p(z)\} > 0$ for $z \in \mathbb{U}$.

Lemma 3. *(See [3].) Let $\kappa, \gamma \in \mathbb{C}$ with $\kappa \neq 0$ and let $h \in H(\mathbb{U})$ with $h(0) = c$. If $\operatorname{Re}\{\kappa h(z) + \gamma\} > 0$ $(z \in \mathbb{U})$, then the solution of the following differential equation:*

$$q(z) + \frac{zq'(z)}{\kappa q(z) + \gamma} = h(z) \quad \big(z \in \mathbb{U};\ q(0) = c\big)$$

is analytic in $\mathbb{U}$ and satisfies $\operatorname{Re}\{\kappa h(z) + \gamma\} > 0$ for $z \in \mathbb{U}$.

Lemma 4. *(See [4].) Let $p \in \mathcal{F}(a)$ and let $q(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$ be analytic in $\mathbb{U}$ with $q(z) \neq a$ and $n \geqslant 1$. If q is not subordinate to p, then there exist two points $z_0 = r_0 e^{i\theta} \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U} \backslash E(q)$ such that:*

$$q(\mathbb{U}_{r_0}) \subset p(\mathbb{U}); \qquad q(z_0) = p(\zeta_0) \quad \text{and} \quad z_0 p'(z_0) = m\zeta_0 p(\zeta_0) \quad (m \geqslant n).$$

Lemma 5. *(See [6].) Let $q \in H[a, 1]$ and $\phi : \mathbb{C}^2 \to \mathbb{C}$. Also set $\phi(q(z), zq'(z)) = h(z)$. If $L(z,t) = \phi(q(z), tzq'(z))$ is a subordination chain and $q \in H[a,1] \cap \mathcal{F}(a)$, then*

$$h(z) \prec \varphi\big(p(z), zp'(z)\big)$$

implies that $q(z) \prec p(z)$. Furthermore, if $\varphi(q(z), zq'(z)) = h(z)$ has a univalent solution $q \in \mathcal{F}(a)$, then q is the best subordinant.

In the present paper, we aim to prove some subordination-preserving and superordination-preserving properties associated with the integral operator $\mathcal{P}_p^\alpha$. Sandwich-type results involving this operator are also derived.

2. Sandwich results involving the operator $\mathcal{P}_p^\alpha$

Unless otherwise mentioned, we assume throughout this section that $\lambda, \mu > 0$, $p \in \mathbb{N}$ and all powers are understood as principal values.

Theorem 1. *Let $f, g \in \sum_p$ and let:*

$$\operatorname{Re}\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta \quad \left(\phi(z) = (1-\gamma)\big(z^p \mathcal{P}_p^\alpha g(z)\big)^\mu + \gamma\left(\frac{\mathcal{P}_p^{\alpha-1} g(z)}{\mathcal{P}_p^\alpha g(z)}\right)\big(z^p \mathcal{P}_p^\alpha g(z)\big)^\mu; z \in \mathbb{U}\right), \tag{2.1}$$

where δ is given by:

$$\delta = \frac{\lambda^2 + \mu^2 - |\lambda^2 - \mu^2|}{4\lambda\mu}. \tag{2.2}$$

Then the subordination condition:

$$(1-\gamma)\big(z^p \mathcal{P}_p^\alpha f(z)\big)^\mu + \gamma\left(\frac{\mathcal{P}_p^{\alpha-1} f(z)}{\mathcal{P}_p^\alpha f(z)}\right)\big(z^p \mathcal{P}_p^\alpha f(z)\big)^\mu \prec (1-\gamma)\big(z^p \mathcal{P}_p^\alpha g(z)\big)^\mu + \gamma\left(\frac{\mathcal{P}_p^{\alpha-1} g(z)}{\mathcal{P}_p^\alpha g(z)}\right)\big(z^p \mathcal{P}_p^\alpha g(z)\big)^\mu$$

implies that:$(z^p \mathcal{P}_p^\alpha f(z))^\mu \prec (z^p \mathcal{P}_p^\alpha g(z))^\mu$ and the function $(z^p \mathcal{P}_p^\alpha g(z))^\mu$ *is the best dominant.*

Proof. Let us define the functions $F(z)$ and $G(z)$ in $\mathbb{U}$ by:

$$F(z) = \big(z^p \mathcal{P}_p^\alpha f(z)\big)^\mu \quad \text{and} \quad G(z) = \big(z^p \mathcal{P}_p^\alpha f(z)\big)^\mu \quad (z \in \mathbb{U}), \tag{2.3}$$

we assume here, without loss of generality, that $G(z)$ is analytic and univalent on $\bar{\mathbb{U}}$ and $G'(\zeta) \neq 0$ $(|\zeta| = 1)$. If not, then we replace $F(z)$ and $G(z)$ by $F(\rho z)$ and $G(\rho z)$, respectively, with $0 < \rho < 1$. These new functions have the desired properties on $\bar{\mathbb{U}}$, and we can use them in the proof of our result. Therefore, the results would follow by letting $\rho \to 1$.

We first show that, if:

$$q(z) = 1 + \frac{zG''(z)}{G'(z)} \quad (z \in \mathbb{U}), \tag{2.4}$$

then $\operatorname{Re}\{q(z)\} > 0$ $(z \in \mathbb{U})$. From (1.6) and the definition of the functions G, ϕ, we obtain that:

$$\phi(z) = G(z) + \frac{\lambda}{\mu} zG'(z). \tag{2.5}$$

Differentiating both side of (2.5) with respect to z yields:

$$\phi'(z) = \left(1 + \frac{\lambda}{\mu}\right)G'(z) + \frac{\lambda}{\mu} zG''(z). \tag{2.6}$$

Combining (2.4) and (2.6), we easily get:

$$1 + \frac{z\phi''(z)}{\phi'(z)} = q(z) + \frac{zq'(z)}{q(z) + \frac{\mu}{\lambda}} = h(z) \quad (z \in \mathbb{U}). \tag{2.7}$$

It follows from (2.1) and (2.7) that:

$$\operatorname{Re}\left\{h(z)+\frac{\mu}{\lambda}\right\}>0 \quad (z\in\mathbb{U}). \tag{2.8}$$

Moreover, by using Lemma 3, we conclude that the differential equation (2.7) has a solution $q(z)\in H(\mathbb{U})$ with $h(0)=q(0)=1$. Let:

$$H(u,v)=u+\frac{v}{u+\frac{\mu}{\lambda}}+\delta,$$

where δ is given by (2.2). From (2.7) and (2.8), we obtain:

$$\operatorname{Re}\left\{H\left(q(z);zq'(z)\right)\right\}>0 \quad (z\in\mathbb{U}).$$

To verify the condition:

$$\operatorname{Re}\left\{H(is;t)\right\}\leqslant 0 \quad \left(s\in\mathbb{R};\, t\leqslant -\frac{1+s^2}{2}\right), \tag{2.9}$$

we proceed as follows:

$$\operatorname{Re}\left\{H(is;t)\right\}=\operatorname{Re}\left\{is+\frac{t}{is+\frac{\mu}{\lambda}}+\delta\right\}=\frac{\frac{\mu}{\lambda}t}{s^2+(\frac{\mu}{\lambda})^2}+\delta\leqslant -\frac{\Psi_p(\lambda,\mu,\delta,s)}{2[s^2+(\frac{\mu}{\lambda\gamma})^2]},$$

where:

$$\Psi_p(\alpha,\beta,\delta,s)=\left[\frac{\mu}{\lambda}-2\delta\right]s^2-2\delta\left(\frac{\mu}{\lambda}\right)^2+\frac{\mu}{\lambda}. \tag{2.10}$$

For δ given by (2.2), we note that the expression $\Psi_p(\lambda,\mu,\delta,s)$ in (2.10) is a positive, which implies that (2.9) holds. Thus, by using Lemma 2, we conclude that: $\operatorname{Re}\{q(z)\}>0$ $(z\in\mathbb{U})$. By the definition of $q(z)$, we know that G is convex. To prove $F\prec G$, let the function $L(z,t)$ be defined by:

$$L(z,t)=G(z)+\frac{(1+t)zG'(z)}{\frac{\mu}{\lambda}} \quad (0\leqslant t<\infty;\, z\in\mathbb{U}). \tag{2.11}$$

Since G is convex, then

$$\left.\frac{\partial L(z,t)}{\partial z}\right|_{z=0}=G'(0)\left(1+\frac{(1+t)}{\frac{\mu}{\lambda}}\right)\neq 0 \quad (0\leqslant t<\infty;\, z\in\mathbb{U})$$

and

$$\operatorname{Re}\left\{\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right\}=\operatorname{Re}\left\{\frac{\mu}{\lambda}+(1+t)q(z)\right\}>0 \quad (0\leqslant t<\infty;\, z\in\mathbb{U}).$$

Therefore, by using Lemma 1, we deduce that $L(z,t)$ is a subordination chain. It follows from the definition of the subordination chain that: $\phi(z)=G(z)+\frac{zG'(z)}{\frac{\mu}{\lambda}}=L(z,0)$, and $L(z,0)\prec L(z,t)$ $(0\leqslant t<\infty)$, which implies that

$$L(\zeta,t)\notin L(\mathbb{U},0)=\phi(\mathbb{U}) \quad (0\leqslant t<\infty;\, \zeta\in\partial\mathbb{U}). \tag{2.12}$$

If F is not subordinate to G, by using Lemma 4, we know that there exist two points $z_0\in\mathbb{U}$ and $\zeta_0\in\partial\mathbb{U}$ such that:

$$F(z_0)=G(\zeta_0) \quad \text{and} \quad z_0F'(z_0)=(1+t)\zeta_0G'(\zeta_0) \quad (0\leqslant t<\infty). \tag{2.13}$$

Hence, by virtue of (1.6) and (2.13), we have:

$$L(\zeta_0,t)=G(\zeta_0)+\frac{(1+t)zG'(\zeta_0)}{\frac{\mu}{\lambda}}=F(z_0)+\frac{z_0F'(z_0)}{\frac{\mu}{\lambda}}$$
$$=(1-\gamma)\left(z_0^p\mathcal{P}_p^\alpha f(z_0)\right)^\mu+\gamma\left(\frac{\mathcal{P}_p^{\alpha-1}f(z_0)}{\mathcal{P}_p^\alpha f(z_0)}\right)\left(z_0^p\mathcal{P}_p^\alpha f(z_0)\right)^\mu\in\phi(\mathbb{U}).$$

This contradicts to the subordination condition of Theorem 1. Thus, we deduce that $F\prec G$. Considering $F=G$, we see that the function G is the best dominant. This completes the proof of Theorem 1. □

We now derive the following superordination result.

Theorem 2. *Let $f, g \in \sum_p$ and let*:

$$\mathrm{Re}\left\{1+\frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta \quad \left(\phi(z)=(1-\gamma)\left(z^p\mathcal{P}_p^\alpha g(z)\right)^\mu+\gamma\left(\frac{\mathcal{P}_p^{\alpha-1}g(z)}{\mathcal{P}_p^\alpha g(z)}\right)\left(z^p\mathcal{P}_p^\alpha g(z)\right)^\mu\right), \tag{2.14}$$

where δ is given by (2.2). *If the function $(1-\gamma)(z^p\mathcal{P}_p^\alpha f(z))^\mu+\gamma(\frac{\mathcal{P}_p^{\alpha-1}f(z)}{\mathcal{P}_p^\alpha f(z)})(z^p\mathcal{P}_p^\alpha f(z))^\mu$ is univalent in $\mathbb{U}$ and $(z^p\mathcal{P}_p^\alpha f(z))^\mu \in \mathcal{F}$, then the superordination condition*:

$$(1-\gamma)\left(z^p\mathcal{P}_p^\alpha g(z)\right)^\mu+\gamma\left(\frac{\mathcal{P}_p^{\alpha-1}g(z)}{\mathcal{P}_p^\alpha g(z)}\right)\left(z^p\mathcal{P}_p^\alpha g(z)\right)^\mu \prec (1-\gamma)\left(z^p\mathcal{P}_p^\alpha f(z)\right)^\mu+\gamma\left(\frac{\mathcal{P}_p^{\alpha-1}f(z)}{\mathcal{P}_p^\alpha f(z)}\right)\left(z^p\mathcal{P}_p^\alpha f(z)\right)^\mu$$

implies that: $(z^p\mathcal{P}_p^\alpha g(z))^\mu \prec (z^p\mathcal{P}_p^\alpha f(z))^\mu$ *and the function* $(z^p\mathcal{P}_p^\alpha g(z))^\mu$ is the best subordinant.

Proof. Suppose that the functions F, G and q are defined by (2.3) and (2.4), respectively. By applying the similar method as in the proof of Theorem 1, we get: $\mathrm{Re}\{q(z)\} > 0$ $(z \in \mathbb{U})$. Next, to arrive at our desired result, we show that $G \prec F$. For this, we suppose that the function $L(z,t)$ is defined is (2.11). Since G is convex, by applying a similar method as in Theorem 1, we deduce that $L(z,t)$ is subordination chain. Therefore, by using Lemma 5, we conclude that $G \prec F$. Moreover, since the differential equation: $\phi(z) = G(z) + \frac{zG'(z)}{\frac{\mu}{\lambda}} = \varphi(G(z), zG'(z))$ has a univalent solution G, it is the best subordinant. This completes the proof of Theorem 2. □

Combining the above-mentioned subordination and superordination results involving the operator $\mathcal{P}_p^\alpha$, the following "sandwich-type result" is derived.

Theorem 3. *Let $f, g_j \in \sum_p$ $(j = 1, 2)$ and let*:

$$\mathrm{Re}\left\{1+\frac{z\phi_j''(z)}{\phi_j'(z)}\right\} > -\delta \quad \left(\phi_j(z)=(1-\gamma)\left(z^p\mathcal{P}_p^\alpha g_j(z)\right)^\mu+\gamma\left(\frac{\mathcal{P}_p^{\alpha-1}g_j(z)}{\mathcal{P}_p^\alpha g_j(z)}\right)\left(z^p\mathcal{P}_p^\alpha g_j(z)\right)^\mu;\ j=1,2;\ z\in\mathbb{U}\right),$$

where δ is given by (2.2). *If the function $(1-\lambda)(z^p\mathcal{P}_p^\alpha f(z))^\mu+\lambda\frac{\mathcal{P}_p^{\alpha-1}f(z)}{\mathcal{P}_p^\alpha f(z)}(z^p\mathcal{P}_p^\alpha f(z))^\mu$ is univalent in $\mathbb{U}$ and $(z^p\mathcal{P}_p^\alpha f(z))^\mu \in \mathcal{F}$, then the condition*:

$$\begin{aligned}
&(1-\lambda)\left(z^p\mathcal{P}_p^\alpha g_1(z)\right)^\mu+\lambda\left(\frac{\mathcal{P}_p^{\alpha-1}g_1(z)}{\mathcal{P}_p^\alpha g_1(z)}\right)\left(z^p\mathcal{P}_p^\alpha g_1(z)\right)^\mu\\
&\quad\prec(1-\lambda)\left(z^p\mathcal{P}_p^\alpha f(z)\right)^\mu+\lambda\left(\frac{\mathcal{P}_p^{\alpha-1}f(z)}{\mathcal{P}_p^\alpha f(z)}\right)\left(z^p\mathcal{P}_p^\alpha f(z)\right)^\mu\\
&\quad\prec(1-\lambda)\left(z^p\mathcal{P}_p^\alpha g_2(z)\right)^\mu+\lambda\left(\frac{\mathcal{P}_p^{\alpha-1}g_2(z)}{\mathcal{P}_p^\alpha g_2(z)}\right)\left(z^p\mathcal{P}_p^\alpha g_2(z)\right)^\mu
\end{aligned}$$

implies that:

$$\left(z^p\mathcal{P}_p^\alpha g_1(z)\right)^\mu \prec \left(z^p\mathcal{P}_p^\alpha f(z)\right)^\mu \prec \left(z^p\mathcal{P}_p^\alpha g_2(z)\right)^\mu$$

and the functions $(z^p\mathcal{P}_p^\alpha g_1(z))^\mu$ and $(z^p\mathcal{P}_p^\alpha g_2(z))^\mu$ are, respectively, the best subordinant and the best dominant.

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