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# Solvability of some quadratic BSDEs without exponential moments ☆

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ABSTRACT

We establish the existence and uniqueness of square integrable solutions for a class of one-dimensional quadratic backward stochastic differential equations (QBSDEs). This is done with a merely square integrable terminal condition, and in some cases with a measurable generator. This shows, in particular, that neither the existence of exponential moments for the terminal condition nor the continuity of the generator are needed for the existence and/or uniqueness of solutions for quadratic BSDEs. These conditions are used in the previous papers on QBSDEs. To do this, we show that Itô's formula remains valid for functions having a merely locally integrable second (generalized) derivative. A comparison theorem is also established.



RÉSUMÉ

Nous établissons l'existence et l'unicité de solutions de carré intégrables pour une classe d'équations différentielles stochastiques rétrogrades (EDSR) quadratiques ayant une condition terminale de carré intégrable, et, dans certains cas, un générateur uniquement mesurable. Le présent travail montre, en particulier, que ni l'existence des moments exponentiels de la donnée terminale, ni la continuité du générateur ne sont nécessaires à l'existence et l'unicité des EDSR quadratiques. Pour ce faire, nous établissons d'abord que, pour les solutions d'EDSR unidimensionnelles de croissance quadratique, la formule d'Itô reste valable pour des fonctions dont la dérivée seconde (au sens des distributions) est seulement localement integrable. Un théorème de comparaison est également établi pour une classe d'EDSR quadratiques ayant un générateur mesurable.



## Version française abrégée

Soit $(Y, Z)$ est une solution de l'EDSR $eq(\xi, H)$, définie dans l'introduction ci-dessous. On suppose que $H$ vérifie l'hypothèse (H2). On montre alors, en utilisant la formule de densité d'occupation du temps, que le temps passé par $Y$ dans un Lebesgue négligeable est négligeable pour la mesure $|Z_t|^2\,dt$. On en déduit une formule du type Itô–Krylov pour les solutions $Y$ appartenant à $\mathcal{S}^2$. C'est-à-dire que $u(Y)$ est une semimartingale d'Itô dès que $u \in \mathcal{C}^1(\mathbb{R}) \cap \mathcal{W}^2_{1,loc}(\mathbb{R})$, où $\mathcal{W}^2_{1,loc}(\mathbb{R})$ désigne l'espace de Sobolev des classes de fonctions localement intégrables dont les dérivées premières et secondes (au sens des distributions) sont également localement intégrables. Pour les résultats suivants (existence, unicité et comparaison des

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solutions), nous supposons que $\xi$ est de carré intégrable, c'est-à-dire que (H1) est vérifiée. Nous appliquons alors cette formule (d'Itô–Krylov) à la fonction $u(y) := \int_0^y \exp(2\int_0^x f(t)\,dt)\,dx$ pour montrer que l'EDSR $eq(\xi, f(y)|z|^2)$ possède une unique solution dans $\mathcal{S}^2 \times \mathcal{M}^2$ dès lors que $f$ est intégrable. En utilisant la même fonction $u$, on montre également que l'EDSR $eq(\xi, a+b|y|+c|z|+f(|y|)|z|^2)$ possède une solution maximale et une solution minimale dans $\mathcal{S}^2 \times \mathcal{M}^2$ pour $f$ intégrable et $a, b, c \in \mathbb{R}$. À noter que l'intégrabilité globale de $f$ oblige cette dernière à ne pas être constante. Cette intégrabilité globale de $f$ confère à la function $u$ de bonnes propriétés globales : en particulier, $u$ est, dans ce cas, une quasi-isométrie. On considère enfin l'EDSR $eq(\xi, H)$ avec $|H(t, y, z)| \leqslant a+b|y|+c|z|+f(|y|)|z|^2$, $f$ intégrable, $H$ et $f$ continus. Nous utilisons alors les solutions des deux EDSR $eq(\xi^-, -(a+b|y|+c|z|+f(|y|)|z|^2))$ et $eq(\xi^+, a+b|y|+c|z|+f(|y|)|z|^2)$ comme barrières ainsi que le résultat de [8] sur les EDSR réfléchies pour montrer que l'EDSR $eq(\xi, H)$ possède une solution dès que l'EDSR dominante $eq(\xi, a+b|y|+c|z|+f(|y|)|z|^2)$ possède une solution. Notre approche permet de couvrir les EDSR de croissance linéaires en posant $f = 0$. Elle donne donc un traitement unifié, pour les EDSR quadratiques et celles de croissance linéaire, en gardant $\xi$ de carré integrable dans les deux situations.

## 1. Introduction

Let $(W_t)_{0\leqslant t\leqslant T}$ be a $d$-dimensional Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $(\mathcal{F}_t)_{0\leqslant t\leqslant T}$ denote the natural filtration of $(W_t)$ such that $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$. Let $\xi$ be an $\mathcal{F}_T$-measurable one-dimensional random variable and $H$ be a real-valued function defined on $[0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d$, which is $(\mathcal{F}_t)$-progressively measurable. The BSDEs with the data $(\xi, H)$ will be referred to as $eq(\xi, H)$. The BSDE under consideration is of the form:

$$Y_t = \xi + \int_t^T H(s, Y_s, Z_s)\,ds - \int_t^T Z_s\,dW_s, \quad 0 \leqslant t \leqslant T. \qquad (eq(\xi, H))$$

We denote $\mathcal{W}^2_{p,loc} := \{u : \mathbb{R} \mapsto \mathbb{R};\ u, u', u'' \in L^p_{loc}(\mathbb{R})\}$.
$\mathcal{S}^2 \times \mathcal{M}^2 := \{(Y, Z), \mathcal{F}_t\text{-adapted and } \mathbb{E}(\sup_{0\leqslant t\leqslant T} |Y_t|^2 + \int_0^T |Z_s|^2\,ds) < \infty\}$.
$\mathcal{L}^p := \{Z, \mathcal{F}_t\text{-adapted and } \int_0^T |Z_s|^p\,ds) < \infty \text{ a.s.}\}$.

**Definition.** A solution of the BSDE $eq(\xi, H)$ is an $\mathcal{F}_t$–adapted process $(Y, Z)$ that satisfies $eq(\xi, H)$ and such that $Y$ is continuous, $\int_0^T |Z_s|^2\,ds$ is finite $\mathbb{P}$-almost surely.

The first aim of this Note consists in showing that Itô's change of variable formula remains valid for $u(Y_t)$ whenever $Y$ is a solution (in $\mathcal{S}^2$) of the quadratic BSDE $eq(\xi, H)$ and $u \in \mathcal{C}^1(\mathbb{R}) \cap \mathcal{W}^2_{1,loc}(\mathbb{R})$. The second aim consists in establishing the existence and uniqueness of solutions for a large class of QBSDEs $eq(\xi, H)$ having a square integrable terminal datum $\xi$ (all the exponential moments of $\xi$ can be infinite) and in many situations the generator $H$ is merely measurable. For instance, the QBSDE $eq(\xi, H)$ has a unique solution in $\mathcal{S}^2 \times \mathcal{M}^2$ in the following situations, which are not covered by the previous works on QBSDEs:

$\xi$ is merely square integrable and $H$ is one of the following generators:
$H_1(y, z) := \sin(y)|z|^2$ if $y \in [-\pi, \frac{\pi}{2}]$ and $H_1(y, z) := 0$ otherwise,
$H_2(y, z) := (\mathbf{1}_{[a,b]}(y) - \mathbf{1}_{[c,d]}(y))|z|^2$ for a given $a < b$ and $c < d$.
$H_3(y, z) := \frac{1}{(1+y^2)\sqrt{|y|}}|z|^2$.

It is worth noting that $H_3(y, z)$ is neither continuous nor locally bounded. Moreover, $H_3$ is not defined for $y = 0$. It should be noted that all the previous papers on QBSDEs are developed in the framework of continuous generators and bounded terminal data or at least having finite exponential moments, see, e.g. [2–5,7–9,11–13].

When the generator $H$ satisfies $|H(s, y, z)| \leqslant a+b|y|+c|z|+f(|y|)|z|^2$ and is continuous in $(y, z)$, our method can be summarized as follows: the existence of solutions for the BSDE $eq(\xi, H)$ can be deduced from the existence of solutions for the QBSDE associated with the dominating generators $a+b|y|+c|z|+f(|y|)|z|^2$. But since the quadratic part of this last is additive, it then can be eliminated by the transformation $u$ (defined above). And hence, the BSDE $eq(\xi, a+b|y|+c|z|+f(|y|)|z|^2)$ can be reduced to a BSDE with linear growth, which is easily solvable. Note finally that in contrast to most previous papers on QBSDEs, our result also covers the BSDEs with linear growth (put $f = 0$). It therefore provides a unified treatment for the quadratic BSDEs and the BSDEs with linear growth, keeping $\xi$ square integrable in both cases.

## 2. Itô–Krylov formula in QBSDEs

We now consider the following assumptions:

(H1) $\xi \in L^2(\Omega)$.
(H2) There exist a positive stochastic process $\eta \in L^1([0,\ T] \times \Omega)$ and a *locally integrable* function $f$ such that, for every $(t, \omega, y, z)$, $|H(t, y, z)| \leqslant \eta_t + |f(y)||z|^2$.

The following Theorem 2.1 and Lemma 2.2 are key elements in our approach. Note that they are interesting in their own and can have potential applications in BSDEs. Note that it is interesting to try to prove Lemma 2.2 by using the methods developed in [10] or [1].

**Theorem 2.1** *(Itô–Krylov's formula for BSDEs). Assume* (H1) *and* (H2) *hold. Let* $(Y, Z)$ *be a solution of BSDE* $eq(\xi, H)$ *in* $\mathcal{S}^2 \times \mathcal{L}^2$. *Assume moreover that* $\int_0^T |H(s, Y_s, Z_s)|\,\mathrm{d}s$ *is finite* $\mathbb{P}$*-almost surely. Then, for any function* $u$ *belonging to* $\mathcal{C}^1(\mathbb{R}) \cap \mathcal{W}^2_{1,loc}(\mathbb{R})$, *we have*:

$$u(Y_t) = u(Y_0) + \int_0^t u'(Y_s)\,\mathrm{d}Y_s + \frac{1}{2}\int_0^t u''(Y_s)|Z_s|\,\mathrm{d}s. \tag{1}$$

**Lemma 2.2** *(Krylov's estimates for BSDEs).*

(i) *Assume that* (H2) *is satisfied. Let* $(Y, Z)$ *be a solution of BSDE* $eq(\xi, H)$ *and* $\tau_R := \inf\{t > 0, |Y_t| \geqslant R\}$. *Assume moreover that* $\int_0^T |H(s, Y_s, Z_s)|\,\mathrm{d}s$ *is finite* $\mathbb{P}$*-almost surely. Then, there exists a positive constant* $C$ *depending on* $T$, $R$ *and* $\|f\|_{L^1([-R,R])}$ *such that for any non-negative measurable function* $\psi$:

$$\mathbb{E}\int_0^{T\wedge\tau_R} \psi(Y_s)|Z_s|^2\,\mathrm{d}s \leqslant C\|\psi\|_{L^1([-R,R])}. \tag{2}$$

(ii) *Assume moreover that* $(Y, Z) \in \mathcal{S}^2 \times \mathcal{L}^2$, (H1) *is satisfied and that*

(H3) *the function* $f$, *appearing in assumption* (H2) *is in* $L^1(\mathbb{R})$.

*Then, for any non-negative measurable function* $\psi$,

$$\mathbb{E}\int_0^T \psi(Y_s)|Z_s|^2\,\mathrm{d}s \leqslant C\|\psi\|_{L^1(\mathbb{R})},$$

*with* $C$ *depending on* $T$, $\|f\|_{L^1(\mathbb{R})}$ *and* $\mathbb{E}(\sup_{0\leqslant t\leqslant T} |Y_t|)$.

**Remark 1.** Using Sobolev's embedding theorem, one can show that formula (1) remains valid for any function $u \in \mathcal{W}^2_{p,loc}(\mathbb{R})$ with $p > 1$.

**Sketch of the proofs.** Lemma 2.2 can be established by applying Tanaka's formula to $(Y - a)^+$ for an arbitrary real number $a$, and by using the time occupation formula. The proof of Theorem 2.1 can be performed by approximating the $\mathcal{C}^1(\mathbb{R}) \cap \mathcal{W}^2_{1,loc}(\mathbb{R})$ class functions by a suitable sequence of $\mathcal{C}^2$ class functions and by using Lemma 2.2. □

We will use Theorem 2.1 to prove the existence and uniqueness of the solution in $\mathcal{S}^2 \times \mathcal{M}^2$ for a class of QBSDEs with a measurable generator.

## 3. QBSDEs with a measurable generator and without exponential moments

According to Dudley [6], there exists a (non necessarily unique) $\mathcal{F}_t$-adapted process $(Z)$ such that $\int_0^T |Z_s|^2\,\mathrm{d}s < \infty$ $\mathbb{P}$-a.s. and $\xi = \int_0^T Z_s\,\mathrm{d}W_s$. The process $(Y)$ defined by $Y_t = \xi - \int_0^t Z_s\,\mathrm{d}W_s$ is $\mathcal{F}_t$-adapted and satisfies the BSDE $eq(\xi, 0)$. This solution $(Y, Z)$ is not unique. But, if we assume $\xi \in L^2(\Omega)$, then the solution is unique and we have $Y_t = \mathbb{E}[\xi/\mathcal{F}_t]$.

**Theorem 3.1.**

(A) *Let* $f$ *be a real-valued integrable function, and assume that* (H1) *is satisfied. Then,*
  (i) *The BSDE* $eq(\xi, f(y)|z|^2)$ *has a unique solution in* $\mathcal{S}^2 \times \mathcal{M}^2$.
  (ii) *For any* $a, b, c \in \mathbb{R}$, *the BSDE* $eq(\xi, a + b|y| + c|z| + f(y)|z|^2)$ *has a minimal and a maximal solutions, which are both in* $\mathcal{S}^2 \times \mathcal{M}^2$.

(B) *Consider now the BSDE* $eq(\xi, H)$. *Assume that* (H1) *and the following conditions are satisfied*:

(H4) $H$ *is continuous in* $(y, z)$, *for a.e.* $(s, \omega)$.

(H5) *There exist positive real numbers* $a$, $b$, $c$ *and a positive continuous integrable function* $f$ *such that, for every* $s$, $y$, $z$:

$$\big|H(s, y, z)\big| \leqslant a + b|y| + c|z| + f\big(|y|\big)|z|^2.$$

*Then, the BSDE* $eq(\xi, H)$ *has at least one solution in* $\mathcal{S}^2 \times \mathcal{M}^2$.

**Sketch of the proofs.** For $f \in L^1(\mathbb{R})$, we define the function

$$u(y) := \int_0^y \exp\left(2\int_0^x f(t)\,\mathrm{d}t\right)\mathrm{d}x. \tag{3}$$

The global integrability condition of the function $f$ gives good properties to the function $u$, in particular the quasi-isometry property (i.e. $u$ and its inverse are uniformly Lipschitz). Now, since $u$ belongs to the space $\mathcal{C}^1(\mathbb{R}) \cap \mathcal{W}^2_{1,loc}(\mathbb{R})$, then applying Theorem 2.1, and using the fact that $u$ is a quasi-isometry, one can prove assertion (A). We shall prove assertion (B).

Put $g(y,z) := a + b|y| + c|z| + f(|y|)|z|^2$. Let $(Y^g, Z^g)$ be the maximal solution of BSDE $eq(\xi^+, g)$. Let $(Y^{-g}, Z^{-g})$ be the minimal solution of BSDE $eq(\xi^-, -g)$. Using the result of Essaky and Hassani [8], Theorem 3.2 with $Y^{-g}$ and $Y^g$ as barriers, we get the existence of a solution to the following reflected BSDE:

$$\left.\begin{array}{ll}
\text{(i)} & Y_t = \xi + \int_t^T H(s, Y_s, Z_s)\,\mathrm{d}s + \int_t^T \mathrm{d}K_s^+ - \int_t^T \mathrm{d}K_s^- - \int_t^T Z_s\,\mathrm{d}B_s, \\
\text{(ii)} & \forall t \leqslant T,\ Y_t^{-g} \leqslant Y_t \leqslant Y_t^g, \\
\text{(iii)} & \int_0^T (Y_s - Y_s^{-g})\,\mathrm{d}K_s^+ = \int_0^T (Y_s^g - Y_s)\,\mathrm{d}K_s^- = 0, \quad \text{a.s.}, \\
\text{(iv)} & K_0^+ = K_0^- = 0, \quad K^+,\ K^-, \text{ are continuous nondecreasing}, \\
\text{(v)} & \mathrm{d}K^+ \perp \mathrm{d}K^-.
\end{array}\right\} \tag{4}$$

Applying Tanaka's formula to $(Y_t^g - Y_t)^+$ and using the fact that $(Y_t^g - Y_t)^+ = (Y_t^g - Y_t)$, one can show that $\mathrm{d}K^- = 0$. Arguing symmetrically, we show that $\mathrm{d}K^+ = 0$. Since, for $K(x) := \int_0^x \exp(-2\int_0^t f(r)\,\mathrm{d}r)\,\mathrm{d}t$, the function $v(|y|) := \int_0^{|y|} K(x)\exp(2\int_0^x f(t)\,\mathrm{d}t)\,\mathrm{d}x$ belongs to $\mathcal{C}^1 \cap \mathcal{W}^2_{1,loc}$, then applying Theorem 2.1 to $v$, one can show that $Z$ belongs to $\mathcal{M}^2$. □

The following proposition allows us to compare the solutions for the BSDEs of type $(eq(\xi, f(y)|z|^2)$. The novelty is that the comparison holds when both the generators are non (necessarily) Lipschitz. Moreover, the generators can merely be compared a.e. in the $y$ variable. The proof is based on Theorem 2.1.

**Proposition 3.1** *(Comparison). Let $\xi_1$, $\xi_2$ be $\mathcal{F}_T$-measurable and satisfy assumption* (H1)*. Let $f$, $g$ be in $L^1(\mathbb{R})$. Let $(Y^f, Z^f)$, $(Y^g, Z^g)$ be respectively the solution of the BSDEs $eq(\xi_1, f(y)|z|^2)$ and $eq(\xi_2, g(y)|z|^2)$. Assume that $\xi_1 \leqslant \xi_2$ a.s. and $f \leqslant g$ a.e. Then $Y_t^f \leqslant Y_t^g$ for all $t$ $\mathbb{P}$-a.s.*

**Remark 2.** We deduce from the previous proposition that for any $\xi$ satisfying (H1) and any integrable functions $f$ and $g$, we have, "*if $f = g$ a.e. then $Y^f = Y^g$*". Here $Y^f$ and $Y^g$ denote the solution the BSDEs $Eq(\xi, f(y)|z|^2)$ and $Eq(\xi, g(y)|z|^2)$, respectively.

**Remark 3.** In assertion (B) of Theorem 3.1, the continuity was imposed to $f$ in order to give an instructive proof which consists in deducing the existence of solutions for a non-reflecting BSDE from the existence of solutions for a reflecting BSDE. Indeed, our proof is based on the result of [8], which requires the continuity of $f$.

**Remark 4.** Assertion (A)-(i) of Theorem 3.1, Proposition 3.1 and Remark 2 allow us to see that there is a gap between the solutions of BSDEs and the viscosity solutions of their associated semilinear PDEs. Indeed, although the BSDE $eq(\xi, f(y)|z|^2)$ has a unique solution, how define the associated PDE (in viscosity sense) when $f$ is defined only a.e.?

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