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Algebraic Geometry

On Cremona transformations of $\mathbb{P}^3$ with all possible bidegrees

Sur les transformations de Cremona de $\mathbb{P}^3$ de tous les degrés possibles

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ABSTRACT

For every orderer pair (d, e) of integer numbers $d, e \geqslant 2$, such that $\sqrt{d} \leqslant e \leqslant d$, we construct a birational map $\mathbb{P}^3 - - \succ \mathbb{P}^3$ defined by homogeneous polynomials of degree d whose inverse map is defined by homogeneous polynomials of degree e.

RÉSUMÉ

Pour chaque paire ordonnée (d, e) d'entiers satisfaisant $d, e \geqslant 2$ et $\sqrt{d} \leqslant e \leqslant d$, nous construisons une application birationnelle $\mathbb{P}^3 - - \succ \mathbb{P}^3$ définie par des formes de degrés d, dont l'application inverse est définie par des formes de degré e.

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1. Introduction

The aim of this note is to correct a mistake in the proof of theorem [4, Théorème 2.2]. The proof of that theorem depends on the example [4, Exemple 2.1] which is wrong.

We propose an explicit construction of Cremona transformations of $\mathbb{P}^3$ (see Section 2, especially Lemma 2) that, together with their inverse maps, provide all possible bidegrees (Theorem 3 and Corollary 4).

2. Main construction and results

Let $\mathbb{P}^3$ be the projective space over an algebraically closed field k of characteristic zero; we fix homogeneous coordinates w, x, y, z on $\mathbb{P}^3$.

We recall that a Cremona transformation of $\mathbb{P}^3$ is a birational map $F : \mathbb{P}^3 - - \succ \mathbb{P}^3$. We say F has *bidegree* (d, e) when F and its inverse F^{-1} are defined by homogeneous polynomials, without non-trivial common factors, of degrees d and e respectively; notice that in this case F^{-1} has bidegree (e, d). If $V \subset \mathbb{P}^3$ is a dense open set over which F^{-1} is defined and injective and $L \subset \mathbb{P}^3$ is a line with $L \cap V \neq \emptyset$, then e is the degree of the closure of $F^{-1}(L \cap V)$; one deduces that $\sqrt{d} \leqslant e \leqslant d$ (see for example [4, §1]).

If $X \subset \mathbb{P}^2$ is a curve and $p \in \mathbb{P}^2$ we denote by $\operatorname{mult}_p(X)$ the multiplicity of X at p. If $S, S' \subset \mathbb{P}^3$ are surfaces and $C \subset S \cap S'$ is an irreducible component, we denote by $\operatorname{mult}_C(S, S')$ the intersection multiplicity of S and S' along C.

Consider a rational map $T : \mathbb{P}^3 - - \succ \mathbb{P}^3$ defined by:

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$$T = (g : qt_1 : qt_2 : qt_3),$$

where $t_1, t_2, t_3 \in k[x, y, z]$ are homogeneous of degree r, without non-trivial common factors, and $g, q \in k[w, x, y, z]$ are homogeneous of degrees d, $d-1$, with $d \geqslant r \geqslant 1$ and g irreducible. We know that T is birational if $\tau := (t_1 : t_2 : t_3) : \mathbb{P}^2 - - \succ \mathbb{P}^2$ is birational and g, q vanish at $o = (1:0:0:0)$ with orders $d-1$ and $\geqslant d-r-1$, respectively (see [3, Proposition 2.2]).

On the other hand, consider $2r-1$ points $p_0, p_1, \ldots, p_{2r-2}$ in $\mathbb{P}^2$, $r \geqslant 2$, satisfying the following condition:

There exist curves $X_r, Y_{r-1} \subset \mathbb{P}^2$ of degrees r, $r-1$, respectively, with X_r irreducible, such that $\mathrm{mult}_{p_0}(X_r) = r-1$, $\mathrm{mult}_{p_0}(Y_{r-1}) \geqslant r-2$ *and $p_i \in X_r \cap Y_{r-1}$ for $i = 1, \ldots, 2r-2$.* (I)

Hence [3] also implies that there exists a plane Cremona transformation defined by polynomials of degree r with a point of multiplicity $r-1$ at p_0 and passing through $p_1, \ldots, p_{2r-2}$ with multiplicity 1: indeed, if we consider $p_0 = (1:0:0)$ and take polynomials t_1 and f, of degrees r and $r-1$, defining X_r and Y_{r-1} respectively, then $(t_1 : yf : zf) : \mathbb{P}^2 - - \succ \mathbb{P}^2$ is a Cremona transformation as required; such a transformation is said to be *associated* with the points $p_0, p_1, \ldots, p_{2r-2}$.

Remark 1. The transformations satisfying condition (I) are general cases of the so-called *de Jonquières transformations* (see [2] or [1, Definition 2.6.10]). We note that the Enriques criterion [1, Theorem 5.1.1] may be used to prove that a set of $2r-2$ points $p_0, p_1, \ldots, p_{2r-2}$ with assigned multiplicities $r-1, 1, \ldots, 1$, and satisfying condition (I) defines a de Jonquières transformation.

Set $r = d$ and take an irreducible homogeneous polynomial $g = wA(x, y, z) + B(x, y, z)$ of degree d; that is, $q \in k - \{0\}$ in the considerations above. Denote by $T_{g,\tau}$ the Cremona transformation defined by:

$$T_{g,\tau} = (g : t_1 : t_2 : t_3), \tag{1}$$

where $\tau = (t_1 : t_2 : t_3)$ is associated to $2d-1$ points satisfying condition (I).

We have:

Lemma 2. *Let $d \geqslant 2$ be an integer number. Then*:

(a) *there exist g and τ such that $T_{g,\tau}$ has bidegree $(d, 2d-1-m)$, for $0 \leqslant m \leqslant d-1$;*
(b) *there exist g and τ such that $T_{g,\tau}$ has bidegree $(d, d^2-\ell^2-m)$, for $0 \leqslant \ell < d-1$ and $0 \leqslant m \leqslant 2d-2$.*

Proof. We identify $\mathbb{P}^2$ with the plane $\{w = 0\} \subset \mathbb{P}^3$ and consider a point $p_0 \in \mathbb{P}^2$. Without loss of generality, we may suppose that $p_0 = (0:1:0:0)$. We recall $o = (1:0:0:0)$.

In order to prove (a) we first choose $g \in k[w, x, y, z]$ to be a homogeneous polynomial that vanishes along the line op_0 with order $d-1$ and is general with respect to this condition. In other words, one has $g = wA + B$ with:

$$A = A_{d-1}(y, z), \qquad B = xB_{d-1}(y, z) + B_d(y, z),$$

where $A_i, B_i \in k[y, z]$ are general homogeneous polynomials of degree i. Hence $A = 0$ defines a union of $d-1$ distinct lines in $\mathbb{P}^2$ passing through p_0 and $B = 0$ defines an irreducible curve of degree d with an ordinary singular point of multiplicity $d-1$ at p_0.

Notice that, by construction, in the open set $\mathbb{P}^2 - \{p_0\}$, curves $A = 0$ and $B = 0$ intersect at $d(d-1) - (d-1)^2 = d-1$ points; in particular, if $m \leqslant d-1$, there exist m points $p_1, \ldots, p_m \in \mathbb{P}^2$ satisfying $A(p_i) = B(p_i) = 0$ for $1 \leqslant i \leqslant m$. We consider m such points and choose $2d-1-m$ points $p_{m+1}, \ldots, p_{2d-2} \in \mathbb{P}^2$ with $A(p_j) \neq 0$ and $B(p_j) = 0$, for all $j = m+1, \ldots, 2d-2$, such that $p_0, p_1, \ldots, p_{2d-2}$ satisfy (I). Let τ be a plane Cremona transformation associated with these $2d-1$ points.

Now we consider a Cremona transformation $T_{g,\tau} : \mathbb{P}^3 - - \succ \mathbb{P}^3$ as in (1). A general member in the linear system defining $T_{g,\tau}$ is an irreducible surface of degree d, S say, with an equation of the form:

$$ag + a_1t_1 + a_2t_2 + a_3t_3 = 0,$$

where $a, a_1, a_2, a_3 \in k$ are general. Therefore, S admits an ordinary singularity of multiplicity $d-1$ at the generic point of (the line) op_0 and is smooth at the generic point of op_i for $1 \leqslant i \leqslant m$. If S' is another general member of that linear system, then there exists an irreducible rational curve Γ of degree $e = \deg(T_{g,\tau}^{-1})$ such that the intersection scheme $S \cap S'$ is supported on:

$$\Gamma \cup \left(\bigcup_{i=0}^{m} op_i\right).$$

We have:

$$\mathrm{mult}_{\Gamma}(S,S')=1,\qquad \mathrm{mult}_{op_0}(S,S')=(d-1)^2,\qquad \mathrm{mult}_{op_i}(S,S')=1,\quad i=1,\dots,m,$$

hence $e=d^2-(d-1)^2-m=2d-1-m$, which proves assertion (a).

To prove (b), we proceed analogously. This time we choose $g=wA+B$ with:

$$A=\sum_{i=\ell}^{d-1}x^{d-1-i}A_i(y,z),\qquad B=\sum_{j=\ell}^{d}x^{d-j}B_j(y,z),$$

where $A_i, B_i\in k[y,z]$ are general homogeneous polynomials of degree i. Since $\ell\leqslant d-2$, there exist points $p_1,\dots,p_{2d-2}\in\mathbb{P}^2$ such that $A(p_i)=B(p_i)=0$ for $1\leqslant i\leqslant m$ and $A(p_j)\neq 0$, $B(p_j)=0$ for $j=m+1,\dots,2d-2$: indeed, in the open set $\mathbb{P}^2-\{p_0\}$, curves $A=0$ and $B=0$ intersect at $d(d-1)-\ell^2\geqslant d(d-1)-(d-2)^2=3d-4$ points. Thus we can define τ as before and obtain assertion (b). □

Theorem 3. *There exist Cremona transformations of bidegree (d,e) for $d\leqslant e\leqslant d^2$.*

Proof. From the part (a) of Lemma 2 we deduce that there exist Cremona transformations of bidegrees (d,e) for $d\leqslant e\leqslant 2d-1$.

Now we use the part (b) of Lemma 2. Suppose $\ell<d-1$ and think of $e=d^2-\ell^2-m$ as a function $e(\ell,m)$ depending on ℓ, m; to complete the proof it suffices to show that the image of that function contains $\{2d,2d+1,\dots,d^2\}$.

We note that $e(d-2,2d-2)=2d-2$ and $e(0,0)=d^2$; in other words, the part (b) of Lemma 2 implies that there exist Cremona transformations of bidegrees $(d,2d-2)$ and (d,d^2). On the other hand $e(\ell,0)-e(\ell-1,2d-2)=2(d-\ell)-1>0$. Since $e(\ell,m)$ decreases with respect to m, we easily obtain the result. □

For $d=2$, the theorem above asserts that there exist Cremona transformations of bidegrees $(2,2),(2,3),(2,4)$; analogously for $d=3$ and bidegrees $(3,3),(3,4),\dots,(3,9)$, and so on. By symmetry, we deduce:

Corollary 4. *There exist Cremona transformations of bidegrees (d,e) with $\sqrt{d}\leqslant e\leqslant d^2$.*

Remark 5. The inequality $\sqrt{d}\leqslant e\leqslant d^2$ is the unique obstruction to the degree for the inverse of a Cremona transformation of degree d in $\mathbb{P}^3$.

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