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Algebraic Geometry

Cubic symmetroids and vector bundles on a quadric surface

Cubiques symétroïdes et fibrés vectoriels sur une surface quadrique

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A R T I C L E   I N F O

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A B S T R A C T

We investigate the jumping conics of stable vector bundles $\mathcal{E}$ of rank 2 on a smooth quadric surface Q with the Chern classes $c_1 = \mathcal{O}_Q(-1,-1)$ and $c_2 = 4$ with respect to the ample line bundle $\mathcal{O}_Q(1,1)$. As a consequence, we prove that the set of jumping conics $S(\mathcal{E})$ uniquely determines $\mathcal{E}$ and that the moduli space of such bundles is rational.

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R É S U M É

Nous étudions les coniques de saut des fibrés vectoriels stables $\mathcal{E}$ de rang 2 sur une surface quadratique lisse Q de classes de Chern $c_1 = \mathcal{O}_Q(-1,-1)$ et $c_2 = 4$ relativement au fibré en droites ample $\mathcal{O}_Q(1,1)$. Nous en déduisons que l'ensemble des coniques de saut $S(\mathcal{E})$ détermine $\mathcal{E}$ de maniére unique et que l'espace de modules de ce type de fibrés est rationnel.

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1. Introduction

Throughout the article, our base field is $\mathbb{C}$, the field of complex numbers.

Let Q be a smooth quadric in $\mathbb{P}_3 = \mathbb{P}(V)$, where V is a 4-dimensional vector space, and $\mathfrak{M}(k)$ be the moduli space of stable vector bundles of rank 2 on Q with the Chern classes $c_1 = \mathcal{O}_Q(-1,-1)$ and $c_2 = k$ with respect to the ample line bundle $\mathcal{L} = \mathcal{O}_Q(1,1)$. $\mathfrak{M}(k)$ forms an open Zariski subset of the projective variety $\overline{\mathfrak{M}}(k)$, whose points correspond to the semi-stable sheaves on Q with the same numerical invariants. The Zariski tangent space of $\mathfrak{M}(k)$ at $\mathcal{E}$ is naturally isomorphic to $H^1(Q, \mathcal{E}nd(\mathcal{E}))$ [8] and so the dimension of $\mathfrak{M}(k)$ is equal to $h^1(Q, \mathcal{E}nd(\mathcal{E})) = 4k-5$, since $\mathcal{E}$ is simple. In [6], we define the jumping conics of $\mathcal{E} \in \mathfrak{M}(k)$ as points in $\mathbb{P}_3^*$ and prove that the set of jumping conic is a symmetric determinantal hypersurface of degree $k-1$ in $\mathbb{P}_3^*$. It enables us to consider a morphism:

$$S : \mathfrak{M}(k) \to \left|\mathcal{O}_{\mathbb{P}_3^*}(k-1)\right| \simeq \mathbb{P}_N.$$

We conjecture in [6] that the general $\mathcal{E} \in \mathfrak{M}(k)$ is uniquely determined by $S(\mathcal{E})$ and prove that this map S is generically injective for $k \leqslant 3$.

In this article, we prove that the conjecture is true when $k = 4$. For $\mathcal{E} \in \mathfrak{M}(4)$, $S(\mathcal{E})$ is a cubic symmetroid surface, i.e. a symmetric determinantal cubic hypersurface in $\mathbb{P}_3^*$. In terms of short exact sequences that $\mathcal{E}$ admits, we can obtain the relation between the singularity of $S(\mathcal{E})$ and the dimension of cohomology of the restriction of $\mathcal{E}$ to its hyperplane section.

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It turns out that $S(\mathcal{E})$ has exactly 4 singular points. It enables us to derive the rationality of $\mathfrak{M}(4)$, which was proven in a much more general setting in [2]. Lastly, we give a brief description of $S(\mathcal{E})$ for non-general bundles of $\mathfrak{M}(4)$. We will denote the dimension of the cohomology $H^i(X,\mathcal{F})$ for a coherent sheaf $\mathcal{F}$ on X by $h^i(X,\mathcal{F})$, or simply by $h^i(\mathcal{F})$ if there is no confusion.

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2. Preliminaries

Let Q be a smooth quadric surface isomorphic to $\mathbb{P}(V_1)\times\mathbb{P}(V_2)$ for two 2-dimensional vector spaces V_1 and V_2. Then Q is embedded into $\mathbb{P}_3\simeq\mathbb{P}(V)$ by the Segre map, where $V=V_1\otimes V_2$. Let us denote $f^*\mathcal{O}_{\mathbb{P}_1}(a)\otimes g^*\mathcal{O}_{\mathbb{P}_1}(b)$ by $\mathcal{O}_Q(a,b)$ and $\mathcal{E}\otimes\mathcal{O}_Q(a,b)$ by $\mathcal{E}(a,b)$ for coherent sheaves $\mathcal{E}$ on Q, where f and g are the projections from Q to each factors. Then the canonical line bundle K_Q of Q is $\mathcal{O}_Q(-2,-2)$. As a direct consequence of the Kunneth formula, we have:

$$H^i\big(Q,\mathcal{O}_Q(a,a+b)\big)=\begin{cases}0, & \text{if } a=-1;\\ H^i(\mathbb{P}_1,\mathcal{O}_{\mathbb{P}_1}(a+b)^{\oplus(a+1)}), & \text{if } a\geqslant 0.\end{cases}$$

Now let us denote the ample line bundle $\mathcal{O}_Q(1,1)$ by $\mathcal{L}$ and let $\overline{\mathfrak{M}}(k)$ be the moduli space of semi-stable sheaves of rank 2 on Q with the Chern classes $c_1=\mathcal{O}_Q(-1,-1)$ and $c_2=k$ with respect to $\mathcal{L}$. The existence and projectivity of $\overline{\mathfrak{M}}(k)$ are shown in [4] and it has an open Zariski subset $\mathfrak{M}(k)$ consisting of the stable vector bundles with the given numeric invariants. By Bogomolov's inequality [8], $\mathfrak{M}(k)$ is empty if $4k<c_1^2=2$ and so we consider only the case of $k\geqslant 1$. The dimension of $\mathfrak{M}(k)$ can be computed to be $h^1(Q,\mathcal{E}nd(\mathcal{E}))=4k-5$. Note that $\mathcal{E}\simeq\mathcal{E}^*(-1,-1)$ and by the Riemann–Roch theorem [5], we have $\chi(\mathcal{E}(m,m))=2m^2+2m+1-k$ for $\mathcal{E}\in\overline{\mathfrak{M}}(k)$. For a hyperplane H in $\mathbb{P}_3$, let $C_H:=Q\cap H$ be the corresponding hyperplane section on Q.

Definition 2.1. The conic $C\subset Q$ is called a *jumping conic* if $h^0(\mathcal{E}|_C)\geqslant 1$.

Remark 2.2. Since any conic $C\subset Q$ is a hyperplane section, we define the set $S(\mathcal{E})$ of jumping conics of $\mathcal{E}$ as a subset of $\mathbb{P}_3^*$. More precisely,

$$S(\mathcal{E}):=\big\{H\in\mathbb{P}_3^*\ \big|\ h^0(\mathcal{E}|_{C_H})\geqslant 1\big\}.$$

When C_H is smooth, it is a jumping conic if the vector bundle $\mathcal{E}$ splits non-generically over it.

Theorem 2.3. *(See [6].) For a Hulsbergen bundle $\mathcal{E}\in\mathfrak{M}(k)$, $S(\mathcal{E})$ is a symmetric determinantal hypersurface of degree $k-1$ in $\mathbb{P}_3^*$ and it has a singular point at $H\in\mathbb{P}_3^*$ if $h^0(\mathcal{E}|_{C_H})\geqslant 2$.*

Remark 2.4. The referee pointed out that the converse might not be true in general. Indeed, the determinant of the following matrix is singular along a line but the ideal of 2×2 minors has length 4:

$$\begin{pmatrix} t_0 & t_1 & t_3\\ t_1 & t_0+t_3 & t_2\\ t_3 & t_2 & 0\end{pmatrix}.$$

Theorem 2.3 enables us to consider a morphism $S:\mathfrak{M}(k)\to|\mathcal{O}_{\mathbb{P}_3^*}(k-1)|\simeq\mathbb{P}_N$ with $N=\binom{k+2}{3}-1$. In [6] and [7], the cases of $k=2,3$ are dealt in detail. For example, when $k=2$, the morphism S extends to an isomorphism from $\overline{\mathfrak{M}}(2)\to\mathbb{P}_3$ and $\mathfrak{M}(2)$ is isomorphic to $\mathbb{P}_3\backslash Q$. In particular, $S(\mathcal{E})$ determines uniquely $\mathcal{E}\in\mathfrak{M}(2)$. A similar result also holds for $k=3$.

3. Results

From now on, we will investigate $S(\mathcal{E})$ for $\mathcal{E}\in\mathfrak{M}(4)$, which is now a *cubic symmetroid surface*, i.e. a symmetric determinantal cubic surface in $\mathbb{P}_3^*$. Note that a nonsingular cubic surface cannot be symmetrically determinantal [3]. Since $\chi(\mathcal{E}(1,1))=1$ and $\mathcal{E}$ is stable, it admits an exact sequence:

$$0\to\mathcal{O}_Q\to\mathcal{E}(1,1)\to\mathcal{I}_Z(1,1)\to 0, \tag{1}$$

where Z is a zero-dimensional subscheme of Q with length 4 and $\mathcal{I}_Z(1,1)$ is the tensor product of the ideal sheaf of Z and $\mathcal{O}_Q(1,1)$. Let us assume that Z is in general position and then we have $h^0(\mathcal{E}(1,1))=1$, which leads us to conclude that for $k=4$, a general $\mathcal{E}$ is a Hulsbergen bundle. In particular, Z is uniquely determined by $\mathcal{E}$. Note that $\mathbb{P}\operatorname{Ext}^1(\mathcal{I}_Z(1,1),\mathcal{O}_Q)\simeq\mathbb{P}H^0(\mathcal{O}_Z)^*\simeq\mathbb{P}_3$. A general point in this family of extensions corresponds to a stable vector bundle [1] and so $\mathfrak{M}(4)$ is birational to a $\mathbb{P}_3$-bundle over the Hilbert scheme $Q^{[4]}$ of zero-dimensional subscheme of Q with length 4. It is consistent with the fact that the dimension of $\mathfrak{M}(4)$ is 11. Note that $Q^{[4]}$ is a resolution of singularity of S^4Q, the fourth symmetric power of Q, and in particular it is 8-dimensional [9].

Assume that Z is not contained in any hyperplane section. If $|Z\cap H|=3$ for a hyperplane section H of $\mathbb{P}_3^*$, we can tensor the sequence (1) with $\mathcal{O}_{C_H}$ to obtain:

$$0\to \mathcal{O}_{C_H}\to \mathcal{E}(1,1)|_{C_H}\to \mathcal{O}_{C_H}(-p)\oplus \mathbb{C}_1\oplus \mathbb{C}_2\oplus \mathbb{C}_3\to 0,$$

where p is a point on C_H. The last surjection gives a surjective map $\mathcal{E}(1,1)|_{C_H}\to \mathcal{O}_{C_H}(-p)$ and its kernel is $\mathcal{O}_{C_H}(3p)$ for degree reason. Twisting by $\mathcal{O}_{C_H}(-2p)$, we obtain:

$$0\to \mathcal{O}_{C_H}(p)\to \mathcal{E}|_{C_H}\to \mathcal{O}_{C_H}(-3p)\to 0.$$

Since $h^0(\mathcal{E}|_{C_H})=2$, H is a singular point of $S(\mathcal{E})$ by Theorem 2.3 and so $S(\mathcal{E})$ has at least 4 singular points.

Proposition 3.1. *For a general vector bundle $\mathcal{E}$ in $\mathfrak{M}(4)$, there are exactly 4 singular points and 6 lines in $S(\mathcal{E})$, i.e. $S(\mathcal{E})$ is a Cayley surface.*

Proof. Similarly as above, we can prove that H is a point of $S(\mathcal{E})$ if $|Z\cap H|=2$, and not a point of $S(\mathcal{E})$ if $|Z\cap H|=1$. Thus the intersection of $S(\mathcal{E})$ with the hyperplane containing a singular point above is the union of three distinct lines, and in particular $S(\mathcal{E})$ contains 6 lines. Let $Z'=\{p_1,\dots,p_4\}\subset S(\mathcal{E})$ be the set of 4 singular points above and denote the line connecting p_i, p_j by l_{ij}. For an arbitrary line $l\subset S(\mathcal{E})$ which is different from l_{ij}, let us assume that l does not intersect with l_{ij}. If $\pi:\mathbb{P}_3^*\dashrightarrow \mathbb{P}_2^*$ is the projection from p_1, then the images of l and l_{ij}, $i,j\neq 1$ intersect. It implies that l and l_{ij} intersect for $i,j\neq 2$. But it is impossible, since the plane containing p_2, p_3, p_4 would contain l. The case of l meeting l_ij can be shown impossible similarly. Thus $S(\mathcal{E})$ contains exactly the 6 lines above and in particular $S(\mathcal{E})$ is not a cone over a plane cubic curve. If $S(\mathcal{E})$ is not normal, then its singular locus would have a 1-dimensional part of degree d and multiplicity m. Its intersection with a generic hyperplane section is a plane cubic curve, and so we have $d=1$ and $m=2$. In other words, the singular locus of $S(\mathcal{E})$ would be a line, which is one of the 6 lines above. It is impossible, since its multiplicity must be 1, and thus $S(\mathcal{E})$ is normal. We can also easily check that $S(\mathcal{E})$ is irreducible, and so the singularities of $S(\mathcal{E})$ are rational double points. Now, without loss of generality, let us assume that $p_1=[1,0,0,0]$ and write the equation f of $S(\mathcal{E})$ by $f=t_0f_2(t_1,t_2,t_3)+f_3(t_1,t_2,t_3)$, where f_i is a homogeneous polynomial of degree i. It is easy to check that if $p=[a_0,a_1,a_2,a_3]\in S(\mathcal{E})$ is a singular point of $S(\mathcal{E})$, then the conic $V(f_2)$ and the cubic $V(f_3)$ intersect at $[a_1,a_2,a_3]$ with multiplicity at least 2. From the irreducibility of $S(\mathcal{E})$, $V(f_2)$ and $V(f_3)$ do not share common components. So the other singular points than p_1 must be contained in the 6 lines above and, by the Bézout theorem, they must be the remaining points in Z'. Hence $S(\mathcal{E})$ contains exactly 4 singular points and 6 lines connecting them. □

Remark 3.2. Considering a $\mathbb{P}_2$-family of hyperplanes of $\mathbb{P}_3$ that contains a point of Z, the intersection of $\mathbb{P}_2$ with $S(\mathcal{E})$ is a cubic plane curve. Since there are 3 hyperplanes in this family, that contain 3 points of Z, so the intersection of the $\mathbb{P}_2$-family with $S(\mathcal{E})$ is the union of three lines.

Conversely, let us consider a cubic hypersurface S_3 in $\mathbb{P}_3^*$ with exactly 4 singular points, say $H_1,\dots,H_4\subset\mathbb{P}_3$. Then H_i's are 4 hyperplanes of $\mathbb{P}_3$ in general position. If S_3 is equal to $S(\mathcal{E})$ for some $\mathcal{E}\in\mathfrak{M}(4)$ with the exact sequence (1), then there are 3 points of Z on each H_i. The intersection of C_{H_1} with H_i, $i=2,3,4$ is two points of Z and so 3 points of Z are determined. The last point is just the intersection of H_2, H_3 and H_4.

Theorem 3.3. *The morphism $S:\mathfrak{M}(4)\to|\mathcal{O}_{\mathbb{P}_3^*}(3)|$ is generically injective. In other words, the set of jumping conics of $\mathcal{E}\in\mathfrak{M}(4)$ uniquely determines $\mathcal{E}$ in general.*

Proof. It is enough to check that for two different stable vector bundles $\mathcal{E}$ and $\mathcal{E}'$ that fit into the sequence (1) with the same Z, $S(\mathcal{E})$ and $S(\mathcal{E}')$ are different. From the previous argument, they have the same singular points. Now, $\mathcal{E}$ and $\mathcal{E}'$ are in the extension family $\mathrm{Ext}^1(\mathcal{I}_Z(1,1),\mathcal{O}_Q)$, which is isomorphic to $H^1(\mathcal{I}_Z(-1,-1))^*$. From the short exact sequence $0\to\mathcal{I}_Z(-1,-1)\to\mathcal{I}_Z\to\mathcal{O}_{C_H}\to 0$, where C_H is a smooth conic that does not intersect with Z, we have:

$$0\to H^1(\mathcal{I}_Z)^*\to H^1\big(\mathcal{I}_Z(-1,-1)\big)^*\stackrel{res}{\to} H^0(\mathcal{O}_{C_H})^*\to 0.$$

Here, the map '*res*' sends $\mathcal{E}$ to $\mathcal{E}|_{C_H}$. Note that $H^1(\mathcal{I}_Z)^*$ is a corank 1-subspace of $H^1(\mathcal{I}_Z(-1,-1))^*$. If we choose H properly so that the image of $H^1(\mathcal{I}_Z)^*$ contains $\mathcal{E}$, but not $\mathcal{E}'$, then their splitting will be different. To be precise, we have $\mathcal{E}|_{C_H}=\mathcal{O}_{C_H}(-2p)\oplus\mathcal{O}_{C_H}$ and $\mathcal{E}'|_{C_H}=\mathcal{O}_{C_H}(-p)^{\oplus 2}$, where p is a point on C_H. In particular, $S(\mathcal{E})$ and $S(\mathcal{E}')$ are different. □

In fact, the argument after Proposition 3.1 can be applied to any symmetric determinantal cubic hypersurface with 4 singular points; we obtain the following:

Corollary 3.4. *$\mathfrak{M}(4)$ is birational to the variety of the symmetric determinantal cubic hypersurfaces $\mathbb{P}_3^*$ with 4 singular points whose corresponding hyperplanes in $\mathbb{P}_3$ satisfy the property that any three hyperplanes among them have the intersection point on Q.*

Proof. It is known in [3] that cubic surfaces with 4 rational double points are projectively isomorphic to the *Cayley 4-nodal cubic surface*, which is a cubic surface with 4 nodal points defined by:

$$t_0t_1t_2 + t_0t_1t_3 + t_0t_2t_3 + t_1t_2t_3 = \det \begin{pmatrix} t_0 & 0 & t_2 \\ 0 & t_1 & -t_2 \\ -t_3 & t_3 & t_2+t_3 \end{pmatrix},$$

which has 4 nodal points $[1,0,0,0]$, $[0,1,0,0]$, $[0,0,1,0]$ and $[0,0,0,1]$. It means that we have a 3-dimensional family of cubic symmetroids for each fixed 4 points as singularities. Here $3 = \dim \mathrm{PGL}(4) - \dim(\mathbb{P}_3^{[4]})$. So the assertion follows automatically from the previous theorem, because the dimension of the variety of the cubic symmetroids in the assertion is $11 = \dim(\mathrm{PGL}(4)) - 4$, which is the dimension of $\mathfrak{M}(4)$. □

Corollary 3.5. *(See Theorem 4.7 in [2].) $\mathfrak{M}(4)$ is rational.*

Proof. Let us prove that the variety Y of the cubic symmetroids with 4 singular points whose corresponding hyperplanes have 4 intersection points on Q is rational. First of all, the variety X of cubic symmetroids with 4 singular points generically has a $\mathbb{P}_3$-bundle structure over $\mathbb{P}_3^{[4]}$ and it is transitively acted by PGL(4). Thus X is rational and we have a dominant map $\pi : X \dashrightarrow \mathbb{P}_3^{[4]}$ to a rational variety $\mathbb{P}_3^{[4]}$. Since Y is a subvariety of X that is generically a $\mathbb{P}_3$-bundle over $Q^{[4]}$ from π and $Q^{[4]}$ is rational, so Y is a rational variety. □

Now let us consider a special case when Z is coplanar. In this case, $S(\mathcal{E})$ is a cubic surface with a unique singular point corresponding to the hyperplane containing Z, say H. Note that $h^0(\mathcal{E}(1,1)) = 2$. Then there is a 1-dimensional family of zero-dimensional subscheme Z for which $\mathcal{E}$ fits into the sequence (1). Such Z should be contained in C_H. For each Z, we can consider the $\mathbb{P}_1$-family of hyperplanes that contain two points of Z, and this corresponds to a line contained in $S(\mathcal{E})$. So we can find 6 lines contained in $S(\mathcal{E})$ out of one such Z. As we vary Z in the 1-dimensional family, we have infinitely many lines through H contained in $S(\mathcal{E})$. Thus we obtain the following statement:

Proposition 3.6. *For the vector bundle $\mathcal{E}$ fitted into the sequence (1) with coplanar Z, $S(\mathcal{E})$ is a cone over a cubic curve in $\mathbb{P}_2^*$ with the vertex point corresponding to the hyperplane containing Z.*

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