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Differential Geometry

A Note on hypersurfaces of a Euclidean space ☆

Une Note sur les hypersurfaces des espaces euclidiens

Sharief Deshmukh

Department of Mathematics, College of Science, King Saud University, P. O. Box-2455, Riyadh 11451, Saudi Arabia

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ABSTRACT

In this short Note, we consider a compact and connected orientable hypersurface M of the Euclidean space R^{n+1} with non-negative support function and Minkowski's integrand σ, and show that the mean curvature function α is the solution of the Poisson equation $\Delta\varphi=\sigma$ if and only if M is isometric to n-sphere $S^n(c)$ of constant curvature c. A similar result is proved for a hypersurface with scalar curvature satisfying the Poisson equation $\Delta\varphi=\sigma$.

RÉSUMÉ

Dans cette courte Note, nous considérons une hypersurface compacte, connexe orientable M de l'espace euclidien R^{n+1}, de fonction support positive ou nulle et d'intégrande de Minkowski σ. Nous montrons que la fonction courbure moyenne α est la solution de l'équation de Poisson $\Delta\varphi=\sigma$ si et seulement si M est isométrique à une sphère $S^n(c)$ de dimension n et courbure constante égale à c. Un résultat similaire est démontré pour une hypersurface de courbure scalaire satisfaisant l'équation de Poisson $\Delta\varphi=\sigma$.

1. Introduction

The importance of the Poisson equation in Physics is well known; it plays a fundamental role in Electrostatics, Fluid motion, and many other areas. On a compact Riemannian manifold (M,g), it is known that the Poisson equation $\Delta\varphi=\rho$ (Δ is the Laplacian operator, ρ is known function, φ is unknown) has a unique solution up to addition of a constant (cf. [1]). It is obvious that the function ρ appearing in the Poisson equation should have an integral equal to 0. Given a compact orientable immersed hypersurface M of the Euclidean space R^{n+1} with support function $\rho=\langle\psi,N\rangle$ and mean curvature function α, the Minkowski integrand $\sigma=1+\rho\alpha$ has an integral equal to zero, where $\psi:M\to R^{n+1}$ is the immersion, N is the unit normal and $\langle,\rangle$ is the Euclidean metric on R^{n+1}. Therefore, it is natural to consider the Poisson equation $\Delta\varphi=\sigma$ on the compact orientable hypersurface M of the Euclidean space R^{n+1}. Characterizing spheres among compact hypersurfaces is one of the fascinating areas in geometry and the use of partial differential equations in characterizing spheres has been recorded in (cf. [2,3]). For the hypersphere $S^n(c)$ in the Euclidean space R^{n+1}, the support function is a positive constant, the Minkowski integrand $\sigma=0$ and the mean curvature α, being a constant, satisfies the Poisson equation $\Delta\varphi=\sigma$. This raises a question: is a compact connected orientable hypersurface of the Euclidean space R^{n+1}, with non-negative support

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E-mail address: shariefd@ksu.edu.sa.

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function and the mean curvature function α satisfying the Poisson equation $\Delta\varphi = \sigma$, necessarily isometric to a sphere $S^n(c)$? In this paper, we answer this question and prove the following:

Theorem 1. *Let M be an orientable compact and connected hypersurface with non-negative support function of the Euclidean space R^{n+1}. The mean curvature function α of the hypersurface M is the solution of the Poisson equation $\Delta\varphi = \sigma$ (σ is the Minkowski integrand) if and only if M is isometric to the n-sphere $S^n(c)$ of constant curvature c.*

Moreover, we also consider the compact and connected orientable hypersurface with certain Ricci curvatures non-negative in the Euclidean space whose scalar curvature is bounded above by the constant $n(n-1)\lambda^{-1}$, where $\lambda = \sup \rho^2$ and show that the scalar curvature of this hypersurface satisfies the Poisson equation $\Delta\varphi = \sigma$ if and only if it is isometric to a sphere $S^n(c)$. Thus, we get another characterization of a sphere in the Euclidean space given by the following:

Theorem 2. *Let M be an orientable compact and connected hypersurface of the Euclidean space R^{n+1} with scalar curvature S bounded above by a constant $n(n-1)\lambda^{-1}$, where $\lambda = \sup \rho^2$, ρ being the support function. Then the Ricci curvature in the direction of the vector field ∇S is non-negative and the scalar curvature S is the solution of the Poisson equation $\Delta\varphi = \sigma$ (σ is the Minkowski integrand) if and only if M is isometric to the n-sphere $S^n(c)$ of constant curvature $c = \lambda^{-1}$.*

2. Preliminaries

Let M be an immersed orientable hypersurface of the Euclidean space R^{n+1} with unit normal vector field N and shape operator A. If $\psi : M \to R^{n+1}$ is the immersion, we denote the induced metric on M by g and by $\langle , \rangle$ the Euclidean metric on R^{n+1}, then we have:

$$\psi = \psi^T + \rho N, \tag{2.1}$$

where $\rho = \langle \psi, N\rangle$ is the support function of the hypersurface M and $\psi^T \in \mathfrak{X}(M)$ the Lie algebra of smooth vector fields on M. Taking covariant derivative in Eq. (2.1) with respect to $X \in \mathfrak{X}(M)$ and using Gauss and Weingarten formulas for a hypersurface, we get:

$$\nabla_X \psi^T = X + \rho AX, \quad \nabla\rho = -A(\psi^T), \; X \in \mathfrak{X}(M), \tag{2.2}$$

where $\nabla\rho$ is the gradient of the support function ρ. If the hypersurface M is compact, the Minkowski formula for the hypersurface is:

$$\int_M (1 + \rho\alpha) = 0, \tag{2.3}$$

where $\alpha = n^{-1} \operatorname{Tr} A$ is the mean curvature of the hypersurface. The shape operator A of the hypersurface satisfies the Codazzi equation:

$$(\nabla A)(X, Y) = (\nabla A)(Y, X), \quad X, Y \in \mathfrak{X}(M),$$

where the covariant derivative $(\nabla A)(X, Y) = \nabla_X AY - A(\nabla_X Y)$. Using local orthonormal frame $\{e_1, \ldots, e_n\}$ on the hypersurface and the above equation, we see that the gradient $\nabla\alpha$ of the mean curvature is given by:

$$n\nabla\alpha = \sum (\nabla A)(e_i, e_i). \tag{2.4}$$

The scalar curvature S of the hypersurface is given by:

$$S = n^2\alpha^2 - \|A\|^2. \tag{2.5}$$

The Minkowski integrand $\sigma = 1 + \rho\alpha$ in Eq. (2.3) gives rise to the Poisson equation:

$$\Delta\varphi = \sigma \tag{2.6}$$

on the hypersurface M. The following result is known for the Poisson equation on a compact Riemannian manifold (M, g).

Theorem 2.1. *(See [1].) On a closed Riemannian manifold (M, g), if σ is a smooth function of integral* 0*, then there is a smooth solution of the equation $\Delta\varphi = \sigma$, unique up to the addition of a constant.*

If φ is a solution of the Poisson equation (2.6), then using:

$$\operatorname{div}(\sigma\nabla\varphi) = g(\nabla\sigma, \nabla\varphi) + \sigma^2 \quad \text{and} \quad \frac{1}{2}\Delta\varphi^2 = \varphi\sigma + \|\nabla\varphi\|^2,$$

we get the following.

Lemma 2.2. *Let M be a compact orientable hypersurface of the Euclidean space R^{n+1} with Minkowski's integrand σ. Then the solution φ of the Poisson equation $\Delta\varphi = \sigma$ satisfies*:

$$\int_M \big(g(\nabla\sigma, \nabla\varphi) + \sigma^2\big) = 0 \quad \text{and} \quad \int_M \big(\varphi\sigma + \|\nabla\varphi\|^2\big) = 0.$$

3. Proof of Theorem 1

Suppose the mean curvature α is the solution of the Poisson equation $\Delta\varphi = \sigma$ on the hypersurface M. Define a smooth function f on M by:

$$f = \frac{1}{2n}\|\psi\|^2. \tag{3.1}$$

Then the gradient of this function is given by $\nabla f = n^{-1}\psi^T$, which together with Eq. (2.2) gives $\Delta f = (1 + \rho\alpha) = \sigma$, that is, f is a solution of the Poisson equation $\Delta\varphi = \sigma$. Hence by Theorem 2.1, we have $\alpha = f + c$, for a constant c and consequently, we get:

$$n\nabla\alpha = \psi^T. \tag{3.2}$$

We denote by A_α the Hessian operator of the mean curvature function α. Then Eqs. (2.2) and (3.2) give:

$$nA_\alpha = I + \rho A,$$

and consequently, we have:

$$n\,\mathrm{Tr}(AA_\alpha) = n\alpha + \rho\|A\|^2. \tag{3.3}$$

We use Eq. (2.4) to compute the divergence of the vector field $A(\nabla\alpha)$:

$$\mathrm{div}\big(A(\nabla\alpha)\big) = \mathrm{Tr}(AA_\alpha) + n\|\nabla\alpha\|^2.$$

Integrating the above equation and using Eq. (3.3), we get:

$$\int_M \big(n\alpha + \rho\|A\|^2 + n^2\|\nabla\alpha\|^2\big) = 0,$$

which together with Lemma 2.2 and $\sigma = 1 + \rho\alpha$ gives:

$$\int_M \big(\rho\big(\|A\|^2 - n\alpha^2\big) + n(n-1)\|\nabla\alpha\|^2\big) = 0.$$

Since the support function ρ is non-negative and $\|A\|^2 \geqslant n\alpha^2$, the above equation gives:

$$\rho\big(\|A\|^2 - n\alpha^2\big) = 0 \quad \text{and} \quad \nabla\alpha = 0.$$

Note that $\rho = 0$ gives a contradiction of the Minkowski formula (2.3). Thus we have $\|A\|^2 - n\alpha^2 = 0$ and α is a constant. However, we know that $\|A\|^2 \geqslant n\alpha^2$ and the equality holds if and only if $A = \alpha I$. Hence, M is totally umbilical hypersurface, which, being compact and connected, is isometric to the n-sphere $S^n(c)$ of constant curvature $c = \alpha^2$. The converse is trivial.

4. Proof of Theorem 2

Suppose M be a compact and connected orientable hypersurface of the Euclidean space R^{n+1} satisfying the hypothesis of the theorem. Then the scalar curvature S satisfies the Poisson equation $\Delta\varphi = \sigma$ and, as we have seen that the function f defined in Eq. (3.1) satisfies the same Poisson equitation, by Theorem 2.1 we have $S = f + c$ for a constant c, which gives:

$$n\nabla S = \nabla f = \psi^T. \tag{4.1}$$

If A_S denotes the Hessian operator of the scalar curvature function S, the above equation together with Eq. (2.2) implies that:

$$nA_S = I + \rho A. \tag{4.2}$$

The above equation and the Minkowski formula (2.3) give:

$$\int_M \|A_S\|^2 = \frac{1}{n^2} \int_M \left(\rho^2 \|A\|^2 - n\right). \tag{4.3}$$

Also, we have $(\Delta S)^2 = \sigma^2 = 1 + 2\rho\alpha + \rho^2\alpha^2$, which on integration gives:

$$\int_M (\Delta S)^2 = \int_M \left(\rho^2\alpha^2 - 1\right), \tag{4.4}$$

where we have used Eq. (2.3). Now, using Eqs. (2.5), (4.3) and (4.4) in the Bochner formula:

$$\int_M \left(Ric(\nabla S, \nabla S) + \|A_S\|^2 - (\Delta S)^2\right) = 0,$$

we get:

$$\int_M \left(Ric(\nabla S, \nabla S) + \frac{1}{n^2}\left(n(n-1) - \rho^2 S\right)\right) = 0. \tag{4.5}$$

Note that the constant $\lambda = \sup \rho^2 > 0$, for if $\lambda = 0$, we shall get $\rho^2 = 0$ and it will give a contradiction of the Minkowski formula. The bound $S \leqslant n(n-1)\lambda^{-1}$ on the scalar curvature gives, $\rho^2 S \leqslant n(n-1)\rho^2\lambda^{-1} \leqslant n(n-1)$. Since the Ricci curvature in the direction of the vector field ∇S is non-negative, Eq. (4.5) gives:

$$Ric(\nabla S, \nabla S) = 0 \quad \text{and} \quad \rho^2 S = n(n-1), \tag{4.6}$$

and the inequality $\rho^2 S \leqslant n(n-1)\rho^2\lambda^{-1} \leqslant n(n-1)$ gives $\rho^2 = \lambda^{-1}$, that is ρ is a constant. Hence, the second equation in (4.6) gives that the scalar curvature S is a constant. Now, using this in Eq. (4.2), we get $A = \rho^{-1} I = \lambda^{-\frac{1}{2}} I$, which proves that M is isometric to $S^n(c)$ of constant curvature $c = \lambda^{-1}$. The converse is trivial.

References

[1] S. Donaldson, Geometric analysis lecture notes, available online at http://www2.imperial.ac.uk/~skdona/.
[2] P. Li, Lecture Notes on Geometric Analysis, Global Analysis Research Center, Seoul National University, Korea, 1993.
[3] P. Li, Curvature and function theory on Riemannian manifolds, in: Surveys in Differential Geometry: Papers Dedicated to Atiyah, Bott, Hirzebruch, and Singer, vol. VII, International Press, 2000, pp. 375–432.