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Partial differential equations

New properties of the Fučík spectrum ☆

Nouveaux résultats sur le spectre de Fučík

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Dedicated to the memory of our Fathers

ABSTRACT

In this Note we present some results on the Fučík spectrum for the Laplace operator, that give new information on its structure. In particular, these results show that, if Ω is a bounded domain of $\mathbb{R}^N$ with $N > 1$, then the Fučík spectrum has infinitely many curves asymptotic to the lines $\{\lambda_1\} \times \mathbb{R}$ and $\mathbb{R} \times \{\lambda_1\}$, where λ_1 denotes the first eigenvalue of the operator $-\Delta$ in $H_0^1(\Omega)$. Notice that the situation is quite different in the case $N = 1$; in fact, in this case, the Fučík spectrum may be obtained by direct computation and one can verify that it includes only two curves asymptotic to these lines. The method we use for the proof is completely variational.

RÉSUMÉ

Dans cette Note, nous présentons des résultats qui donnent de nouvelles informations sur la structure du spectre de Fučík pour l'operateur de Laplace. En particulier, ces résultats montrent que, si Ω est un domaine borné de $\mathbb{R}^N$ avec $N > 1$, alors le spectre de Fučík a un nombre infini de courbes qui ont comme asymptotes les droites $\{\lambda_1\} \times \mathbb{R}$ et $\mathbb{R} \times \{\lambda_1\}$, où λ_1 est la première valeur propre de l'operateur $-\Delta$ in $H_0^1(\Omega)$. La situation est bien différente dans le cas $N = 1$; en effect, dans ce cas, on peut vérifier qu'il y a seulement deux courbes dans le spectre de Fučík, qui ont ces droites comme asymptotes. La méthode de démonstration que nous avons suivie est complètement variationnelle.

Version française abrégée

Dans cette Note, nous présentons des resultats nouveaux (qui sont prouvés dans [7]) sur la structure du spectre de Fučík, qui est important dans l'etude de certaines equations elliptiques (voir, par exemple, [1,3,2,4–6], etc.).

Soit Ω un domaine borné de $\mathbb{R}^N$; le spectre de Fučík pour l'operateur de Laplace $-\Delta$ in $H_0^1(\Omega)$ est défini comme l'ensemble Σ de toutes les paires $(\alpha, \beta) \in \mathbb{R}^2$ telles que le problème

$$\Delta u - \alpha u^- + \beta u^+ = 0 \quad \text{dans } \Omega, \qquad u = 0 \quad \text{sur } \partial\Omega \tag{1}$$

a des solutions $u \not\equiv 0$.

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Si $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \cdots$ sont les valeurs propres de $-\Delta$ dans $H_0^1(\Omega)$, il est évident que Σ comprend les droites $\{\lambda_1\} \times \mathbb{R}$, $\mathbb{R} \times \{\lambda_1\}$ et toutes les paires (λ_i, λ_i) $\forall i \in \mathbb{N}$. Si $N = 1$, il y a seulement deux courbes dans Σ, qui ont les droites $\{\lambda_1\} \times \mathbb{R}$ et $\mathbb{R} \times \{\lambda_1\}$ comme asymptotes (elles proviennent, respectivement, de (λ_2, λ_2) et de (λ_3, λ_3)). Au contraire, si $N > 1$, les resultats présentés dans cette Note montrent que, dans Σ, il y a un nombre infini de courbes, qui ont les droites $\{\lambda_1\} \times \mathbb{R}$ et $\mathbb{R} \times \{\lambda_1\}$ comme asymptotes.

Les résultats principaux sont rassemblés dans le théorème suivant :

Théorème 0.1. *Soit Ω un domaine borné, régulier et connexe de $\mathbb{R}^N$ avec $N \geqslant 2$. Alors, pour tout $k \in \mathbb{N}$, il existe $\beta_k > 0$, qui a les proprietés suivantes. Pour tout $\beta > \beta_k$, il existe $\alpha_{k,\beta} > \lambda_1$ et $u_{k,\beta} \in H_0^1(\Omega)$, avec $u_{k,\beta}^+ \not\equiv 0$ et $u_{k,\beta}^- \not\equiv 0$, telles que (1), avec $\alpha = \alpha_{k,\beta}$ et $u = u_{k,\beta}$, est satisfait pour tout $\beta > \beta_k$. De plus, pour tout $k \in \mathbb{N}$, $\alpha_{k,\beta}$ est une fonction continue de β et*

(a) $\alpha_{k,\beta} < \alpha_{k+1,\beta}$ $\forall \beta > \max\{\beta_k, \beta_{k+1}\}$;

(b) *quand $\beta \to +\infty$, $\alpha_{k,\beta} \to \lambda_1$ et $u_{k,\beta} \to -e_1$ in $H_0^1(\Omega)$, où e_1 est la fonction positive de $H_0^1(\Omega)$, telle que $\Delta e_1 + \lambda_1 e_1 = 0$ in Ω et $\int_\Omega e_1^2 \, dx = 1$.*

(c) *Il existe $r > 0$ tel que, pour tout $\beta > \beta_k$, on peut choisir k points $x_{1,\beta}, \ldots, x_{k,\beta}$ dans Ω, qui ont les propriétés suivantes :*
  (1) $\operatorname{dist}(x_{i,\beta}, \partial\Omega) > r/\sqrt{\beta}$ $\forall i \in \{1, \ldots, k\}$, $|x_{i,\beta} - x_{j,\beta}| > 2r/\sqrt{\beta}$ *si* $i \neq j$;
  (2) $u_{k,\beta}(x) \leqslant 0$ $\forall x \in \Omega \setminus \bigcup_{i=1}^k B(x_{i,\beta}, \frac{r}{\sqrt{\beta}})$ *et* $u_{k,\beta}^+ \not\equiv 0$ *dans* $B(x_{i,\beta}, \frac{r}{\sqrt{\beta}})$ $\forall i \in \{1, \ldots, k\}$;
  (3) $\lim_{\beta \to +\infty} e_1(x_{i,\beta}) = \max_\Omega e_1$ $\forall i \in \{1, \ldots, k\}$, $\lim_{\beta \to +\infty} \sqrt{\beta}|x_{i,\beta} - x_{j,\beta}| = \infty$ *si* $i \neq j$;
  (4) *si* $\rho_\beta > 0$ *et* $\lim_{\beta \to +\infty}(\rho_\beta \sqrt{\beta}) = \infty$, *alors*

$$\lim_{\beta \to +\infty} \sup\left\{ \left|u_{k,\beta}(x) + e_1(x)\right| : x \in \Omega \setminus \bigcup_{i=1}^k B(x_{i,\beta}, \rho_\beta) \right\} = 0;$$

  (5) *si, $\forall \beta > \beta_k$, on définit $s_{i,k,\beta} = \sup\{u_{k,\beta}(x): x \in B(x_{i,\beta}, r/\sqrt{\beta})\}$ et $U_{i,k,\beta}(x) = \frac{1}{s_{i,k,\beta}} u_{k,\beta}(\frac{x}{\sqrt{\beta}} + x_{i,\beta})$ $\forall x \in \sqrt{\beta}(\Omega - x_{i,\beta})$, alors on a : $\lim_{\beta \to +\infty} U_{i,k,\beta}(x) = U(x)$ $\forall x \in \mathbb{R}^N$, $\forall k \in \mathbb{N}$, $\forall i \in \{1, \ldots, k\}$, où U est la solution radiale du problème :*

$$\Delta U + U^+ = 0 \quad \text{dans } \mathbb{R}^N, \qquad U(0) = 1.$$

La difference entre les cas $N = 1$ et $N > 1$ est encore plus évident si on remplace la condition de Dirichlet $u = 0$ par la condition de Neumann $\frac{\partial u}{\partial \nu} = 0$ sur $\partial\Omega$. En effet, si $\tilde\lambda_1 < \tilde\lambda_2 \leqslant \tilde\lambda_3 \leqslant \cdots$ et $\tilde\Sigma$ sont, respectivement, les valeur propres et le spectre de Fučík pour l'operateur $-\Delta$ avec la condition de Neumann sur $\partial\Omega$, alors la première valeur propre est $\tilde\lambda_1 = 0$, les droites $\{0\} \times \mathbb{R}$ et $\mathbb{R} \times \{0\}$ sont contenues dans $\tilde\Sigma$ et, si $N = 1$, il n'y a aucune courbe contenue dans $\tilde\Sigma$ qui ait les droites $\{0\} \times \mathbb{R}$ et $\mathbb{R} \times \{0\}$ comme asymptotes ; au contraire, si $N > 1$, les resultats presentés dans cette Note montrent que, dans $\tilde\Sigma$, il y a un nombre infini de courbes qui ont ces droites comme asymptotes.

La méthode de démonstration est complètement variationnelle ; pour tout $\beta > 0$, la solution u est obtenue comme point critique de la fonctionnelle $f_\beta(u) = \int_\Omega [|Du|^2 - \beta(u^+)^2] \, dx$, definie sur $V = \{u \in H^1(\Omega): \int_\Omega (u^-)^2 \, dx = 1\}$ (tandis que α est obtenu comme multiplicateur de Lagrange).

1. Introduction

Let us consider the Dirichlet problem:

$$\Delta u + g(x, u) = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a smooth bounded connected domain of $\mathbb{R}^N$ and g is a Carathéodory function in $\Omega \times \mathbb{R}$ such that

$$\lim_{t \to -\infty} \frac{g(x,t)}{t} = \alpha, \qquad \lim_{t \to +\infty} \frac{g(x,t)}{t} = \beta \quad \forall x \in \Omega,$$

with α and β in $\mathbb{R}$. This problem may lack of compactness in the sense that the well-known Palais–Smale compactness condition fails if the pair (α, β) belongs to the Fučík spectrum Σ, defined as the set of all the pairs $(\alpha, \beta) \in \mathbb{R}^2$ such that the Dirichlet problem

$$\Delta u - \alpha u^- + \beta u^+ = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \tag{2}$$

has nontrivial solutions ($u \in H_0^1(\Omega)$, $u \not\equiv 0$).

After the first papers [1–3], many researches have been devoted to study problems of this type; so now there exists a very extensive literature on this subject (see, for example, [4–6] and the references therein).

In [4–6], we obtained solutions of problems of this type using a new variational method that does not require to know whether or not $(\alpha, \beta) \in \Sigma$ and, in addition, may be useful to obtain more information on the structure of Σ.

Let us denote by $\lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots$ the eigenvalues of $-\Delta$ in $H_0^1(\Omega)$; obviously, Σ contains all the pairs (λ_i, λ_i) $\forall i \in \mathbb{N}$, that are the unique pairs (α, β) of Σ such that $\alpha = \beta$, and includes the lines $\{\lambda_1\} \times \mathbb{R}$ and $\mathbb{R} \times \{\lambda_1\}$ (which are isolated in Σ, as proved in [2]); if $\alpha \neq \lambda_1$, $\beta \neq \lambda_1$ and $(\alpha,\beta) \in \Sigma$, then $\alpha > \lambda_1$, $\beta > \lambda_1$ and the Fučík eigenfunctions u corresponding to (α,β) are sign-changing functions; moreover, $(\alpha,\beta) \in \Sigma$ if and only if $(\beta,\alpha) \in \Sigma$ because a function u solves (2) if and only if $-u$ solves (2) with (β,α) in place of (α,β).

If $N = 1$, Σ may be obtained by direct computation. It consists of curves emanating from the pairs (λ_i,λ_i) $\forall i \in \mathbb{N}$; if i is an even positive integer, there exists only one curve, while, if i is odd, there exist exactly two curves emanating from (λ_i,λ_i). All these curves are smooth, unbounded and decreasing (i.e., on each curve, α decreases as β increases); moreover, on each curve, α tends to an eigenvalue of $-\Delta$ in $H_0^1(\Omega)$ as $\beta \to +\infty$; conversely, for every eigenvalue λ_i, there exist exactly three curves asymptotic to the lines $\{\lambda_i\} \times \mathbb{R}$ and $\mathbb{R} \times \{\lambda_i\}$: they pass, respectively, through the pairs $(\lambda_{2i-1},\lambda_{2i-1})$, $(\lambda_{2i},\lambda_{2i})$ and $(\lambda_{2i+1},\lambda_{2i+1})$. In particular, if $N = 1$, there are only two nontrivial curves of Σ asymptotic to $\{\lambda_1\} \times \mathbb{R}$ and $\mathbb{R} \times \{\lambda_1\}$.

The situation is quite different in the case $N > 1$. In fact, the results we present in this Note (which are proved in [7] using the method developed in [4–6]) show that, if $N > 1$, there exist infinitely many curves of the Fučík spectrum Σ, asymptotic to the lines $\{\lambda_1\} \times \mathbb{R}$ and $\mathbb{R} \times \{\lambda_1\}$.

The difference between the cases $N = 1$ and $N > 1$ becomes even more evident if in (2) we replace the Dirichlet boundary condition by the Neumann condition $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$.

Let us denote by $0 = \tilde{\lambda}_1 < \tilde{\lambda}_2 \leqslant \tilde{\lambda}_3 \leqslant \cdots$ and by $\tilde{\Sigma}$, respectively, the eigenvalues of $-\Delta$ and the Fučík spectrum with the Neumann boundary condition. If $N = 1$, a direct computation shows that no curve of the Fučík spectrum $\tilde{\Sigma}$ is asymptotic to the lines $\{0\} \times \mathbb{R}$ and $\mathbb{R} \times \{0\}$. On the contrary, if $N > 1$, the results presented in this Note show that in $\tilde{\Sigma}$ there exist infinitely many curves asymptotic to the lines $\{0\} \times \mathbb{R}$ and $\mathbb{R} \times \{0\}$.

The method we use for the proof is completely variational. For all $\beta > 0$, we consider the functional f_β defined by $f_\beta(u) = \int_\Omega [|Du|^2 - \beta(u^+)^2]\,dx$, constrained on the set $V = \{u \in H^1(\Omega)\colon \int_\Omega (u^-)^2\,dx = 1\}$. The functional f_β is defined in $H_0^1(\Omega)$ or in $H^1(\Omega)$, respectively, in the case of Dirichlet or Neumann boundary conditions. In both cases, the Fučík eigenfunction u is obtained as constrained critical point for f_β on V, for all $\beta > 0$, while α arises as the Lagrange multiplier with respect to the constraint V. The Fučík eigenfunction u we obtain in this way, corresponding to the pair (α,β), presents a prescribed number of bumps; this number may be fixed in an arbitrary way and it remains constant on each curve of the Fučík spectrum that we construct using this method. Finally, notice that this method allows us also to describe in a natural way the asymptotic behaviour, as α or β tend to $+\infty$, of the obtained eigenfunction u; in particular, we show that the bumps of u concentrate near suitable points (of Ω or of $\partial\Omega$) and we describe the asymptotic profile of the rescaled bumps.

2. The main results

Let us first consider the Fučík spectrum in the case of Dirichlet boundary conditions.

Theorem 2.1. *Let Ω be a smooth bounded connected domain of $\mathbb{R}^N$ with $N \geqslant 2$. Then, for every positive integer k, there exists $\beta_k > 0$ having the following properties. For all $\beta > \beta_k$, there exist $\alpha_{k,\beta} > \lambda_1$ and $u_{k,\beta} \in H_0^1(\Omega)$, with $u_{k,\beta}^+ \not\equiv 0$ and $u_{k,\beta}^- \not\equiv 0$, such that (2), with $\alpha = \alpha_{k,\beta}$ and $u = u_{k,\beta}$, is satisfied for all $\beta > \beta_k$. Moreover, $\alpha_{k,\beta}$ depends continuously on β for all $k \in \mathbb{N}$, $\alpha_{k,\beta} < \alpha_{k+1,\beta}$ $\forall \beta > \max\{\beta_k,\beta_{k+1}\}$, $\alpha_{k,\beta} \to \lambda_1$, as $\beta \to +\infty$, while $u_{k,\beta} \to -e_1$ in $H_0^1(\Omega)$, where e_1 is the positive function in $H_0^1(\Omega)$ such that $\Delta e_1 + \lambda_1 e_1 = 0$ in Ω and $\int_\Omega e_1^2\,dx = 1$.*

In addition, there exists $r > 0$ such that, for all $\beta > \beta_k$, there exist k points $x_{1,\beta},\dots,x_{k,\beta}$ in Ω, having the following properties: $\operatorname{dist}(x_{i,\beta},\partial\Omega) > r/\sqrt{\beta}$ for $i = 1,\dots,k$, $|x_{i,\beta} - x_{j,\beta}| > 2r/\sqrt{\beta}$ for $i \neq j$, $u_{k,\beta}(x) \leqslant 0$ $\forall x \in \Omega \setminus \bigcup_{i=1}^k B(x_{i,\beta}, r/\sqrt{\beta})$ and $u_{k,\beta}^+ \not\equiv 0$ in $B(x_{i,\beta}, r/\sqrt{\beta})$ for $i = 1,\dots,k$. Moreover,

$$\lim_{\beta\to+\infty} e_1(x_{i,\beta}) = \max_\Omega e_1, \quad \text{for } i = 1,\dots,k, \quad \text{and} \quad \lim_{\beta\to+\infty} \sqrt{\beta}|x_{i,\beta} - x_{j,\beta}| = \infty \quad \text{for } i \neq j.$$

If $\rho_\beta > 0$ $\forall \beta > \beta_k$ and $\lim_{\beta\to+\infty}(\rho_\beta\sqrt{\beta}) = \infty$, then $\lim_{\beta\to+\infty} \sup\{|u_{k,\beta}(x) + e_1(x)|\colon x \in \Omega \setminus \bigcup_{i=1}^k B(x_{i,\beta},\rho_\beta)\} = 0$.

If, $\forall k \in \mathbb{N}$, $\forall i \in \{1,\dots,k\}$, $\forall \beta > \beta_k$ and $\forall x \in \sqrt{\beta}(\Omega - x_{i,\beta})$ we set $U_{i,k,\beta}(x) = (1/s_{i,k,\beta})u_{k,\beta}(x/\sqrt{\beta} + x_{i,\beta})$, where $s_{i,k,\beta} = \sup\{u_{k,\beta}(x)\colon x \in B(x_{i,\beta}, r/\sqrt{\beta})\}$, then the rescaled function $U_{i,k,\beta}$ converges as $\beta \to +\infty$ to the radial solution U of the problem:

$$\Delta U + U^+ = 0 \quad \text{in } \mathbb{R}^N, \qquad U(0) = 1 \tag{3}$$

and the convergence is uniform on the compact subsets of $\mathbb{R}^N$.

Let us point out that the asymptotic behaviour of $u_{k,\beta}$, as $\beta \to +\infty$, is different in the cases $N = 2$ and $N > 2$. In fact, if $N = 2$, we have $\lim_{\beta\to+\infty} s_{i,k,\beta} = 0$ for every $k \in \mathbb{N}$ and $i \in \{1,\dots,k\}$ while, if $N > 2$, $\lim_{\beta\to+\infty} s_{i,k,\beta} = c$, where c is a positive constant depending only on N and $\sup_\Omega e_1$. This different behaviour is strictly related to the fact that, if U is the radial solution of problem (3), then $\inf_{\mathbb{R}^N} U = -\infty$ for $N = 2$, while $\inf_{\mathbb{R}^N} U > -\infty$ for $N > 2$.

Remark 1. The eigenfunction $u_{k,\beta}$ given by Theorem 2.1 presents k bumps that, as $\beta \to +\infty$, concentrate near the points $x_{1,\beta},\dots,x_{k,\beta}$, approaching maximum points of e_1; moreover, if the distance between two concentration points tends to zero as $\beta \to +\infty$, then the approaching rate is less than the concentration rate, so that the bumps remain quite distinct. Notice that one can construct eigenfunctions of different types, with k bumps localized near points of the boundary of Ω, that also give rise to infinitely many curves of the Fučík spectrum, asymptotic to the lines $\{\lambda_1\}\times\mathbb{R}$ and $\mathbb{R}\times\{\lambda_1\}$. In fact, under the assumptions of Theorem 2.1, for every positive integer k there exists $\bar\beta_k>0$, having the following properties. For all $\beta>\bar\beta_k$, there exists $\bar\alpha_{k,\beta}>\lambda_1$ and $v_{k,\beta}\in H^1_0(\Omega)$, with $v^+_{k,\beta}\not\equiv 0$ and $v^-_{k,\beta}\not\equiv 0$, such that (2), with $\alpha=\bar\alpha_{k,\beta}$ and $u=v_{k,\beta}$, is satisfied for all $\beta>\bar\beta_k$. Moreover, for all $k\in\mathbb{N}$, $\bar\alpha_{k,\beta}$ depends continuously on β, $\bar\alpha_{k,\beta}<\bar\alpha_{k+1,\beta}$ $\forall\beta>\max\{\bar\beta_k,\bar\beta_{k+1}\}$ and $\bar\alpha_{k,\beta}\to\lambda_1$, as $\beta\to+\infty$, while $v_{k,\beta}\to -e_1$ in $H^1_0(\Omega)$. Furthermore, the k bumps of $v_{k,\beta}$ concentrate, as $\beta\to+\infty$, near k points approaching the boundary of Ω, and the concentration rate is greater than the approaching rates between two distinct concentration points or between the concentration points and the boundary.

Notice that, if the boundary of Ω consists of more than one connected component, then one can construct eigenfunctions $v_{k,\beta}$ as in Remark 1, but with the bumps localized near prescribed connected components of $\partial\Omega$. In analogous way, if for example there exists an open subset A of Ω such that $\sup_{\partial A}e_1<\sup_A e_1$, then one can construct solutions $u_{k,\beta}$ as in Theorem 2.1, with the k bumps localized near k concentration points $x_{1,\beta},\dots,x_{k,\beta}$, with rescaled bumps having the same asymptotic profile (still described by the radial solution U of (3)), but with the concentration points that, as $\beta\to+\infty$, approach maximum points of e_1 in A (i.e. $x_{i,\beta}\to x_i$ as $\beta\to+\infty$, with $x_i\in A$ and $e_1(x_i)=\max_A e_1$ for $i=1,\dots,k$).

Now, let us describe the properties of the Fučík spectrum in the case of Neumann boundary conditions. Let us consider the Neumann problem:

$$\Delta u-\alpha u^-+\beta u^+=0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial\nu}=0 \quad \text{on } \partial\Omega. \tag{4}$$

If $N=1$, the Fučík spectrum can be obtained by direct computation. In this case we have $0=\tilde\lambda_1<\tilde\lambda_2<\tilde\lambda_3<\cdots$ and the Fučík spectrum consists of the lines $\{0\}\times\mathbb{R}$ and $\mathbb{R}\times\{0\}$ and of infinitely many curves $C_2,C_3,\dots$ having the following properties: for every $i\geqslant 2$, C_i is a smooth, unbounded, decreasing curve, emanating from $(\tilde\lambda_i,\tilde\lambda_i)$ and asymptotic to the lines $\{\tilde\lambda_i/4\}\times\mathbb{R}$ and $\mathbb{R}\times\{\tilde\lambda_i/4\}$ (notice that $\tilde\lambda_i/4$ is an eigenvalue of $-\Delta$ in $H^1(\Omega)$ if and only if i is an odd positive integer and, in this case, $\tilde\lambda_i/4=\tilde\lambda_{(i+1)/2}$). Therefore, if $N=1$, no curve is asymptotic to the lines $\{0\}\times\mathbb{R}$ and $\mathbb{R}\times\{0\}$ and every nontrivial pair (α,β) of the Fučík spectrum satisfies $\alpha>\tilde\lambda_2/4$ and $\beta>\tilde\lambda_2/4$, with $\tilde\lambda_2>0$.

The situation is quite different if $N>1$, as shown by the following theorem.

Theorem 2.2. *Let Ω be a smooth bounded domain of $\mathbb{R}^N$ with $N\geqslant 2$. Then, for every positive integer k, there exists $\tilde\beta_k>0$ having the following properties. For all $\beta>\tilde\beta_k$ there exists $\tilde\alpha_{k,\beta}>0$ and $\tilde u_{k,\beta}\in H^1(\Omega)$, with $\tilde u^+_{k,\beta}\not\equiv 0$ and $\tilde u^-_{k,\beta}\not\equiv 0$, such that (4), with $\alpha=\tilde\alpha_{k,\beta}$ and $u=\tilde u_{k,\beta}$, is satisfied for all $\beta>\tilde\beta_k$. Moreover, for all $k\in\mathbb{N}$, $\tilde\alpha_{k,\beta}$ depends continuously on β, $\tilde\alpha_{k,\beta}<\tilde\alpha_{k+1,\beta}$ $\forall\beta>\max\{\tilde\beta_k,\tilde\beta_{k+1}\}$, $\tilde\alpha_{k,\beta}\to 0$, as $\beta\to+\infty$, while $\tilde u_{k,\beta}\to-[\text{meas}(\Omega)]^{-\frac12}$ in $H^1(\Omega)$.*

For all $\beta>\tilde\beta_k$, there exist k points $\tilde x_{1,\beta},\dots,\tilde x_{k,\beta}$ in Ω, having the following properties:

(1) $\lim_{\beta\to+\infty}\sqrt\beta\,\text{dist}(\tilde x_{i,\beta},\partial\Omega)=\infty$ *for* $i=1,\dots,k$, $\lim_{\beta\to+\infty}\sqrt\beta|\tilde x_{i,\beta}-\tilde x_{j,\beta}|=\infty$ *for* $i\neq j$;
(2) *there exists a positive constant $\tilde r$ such that $\tilde u_{k,\beta}(x)\leqslant 0$ $\forall x\in\Omega\setminus\bigcup_{i=1}^k B(\tilde x_{i,\beta},\tilde r/\sqrt\beta)$ and $\sup\{\tilde u_{k,\beta}(x)\colon x\in B(\tilde x_{i,\beta},\tilde r/\sqrt\beta)\}>0$ for $i=1,\dots,k$, $\forall\beta>\tilde\beta_k$;*
(3) *if $\rho_\beta>0$ $\forall\beta>\tilde\beta_k$ and $\lim_{\beta\to+\infty}(\rho_\beta\sqrt\beta)=\infty$, then $\lim_{\beta\to+\infty}\sup\{|\tilde u_{k,\beta}(x)+[\text{meas}(\Omega)]^{-1/2}|\colon x\in\Omega\setminus\bigcup_{i=1}^k B(\tilde x_{i,\beta},\rho_\beta)\}=0$.*

Moreover, if we set $\tilde s_{i,k,\beta}=\sup\{\tilde u_{k,\beta}(x)\colon x\in B(\tilde x_{i,\beta},\tilde r/\sqrt\beta)\}$ and $\tilde U_{i,k,\beta}(x)=(1/\tilde s_{i,k,\beta})\tilde u_{k,\beta}(x/\sqrt\beta+\tilde x_{i,\beta})$ $\forall x\in\sqrt\beta(\Omega-\tilde x_{i,\beta})$, then:

(a) *if $N=2$, we have $\lim_{\beta\to+\infty}\tilde s_{i,k,\beta}=0$ $\forall k\in\mathbb{N}$, $\forall i\in\{1,\dots,k\}$;*
(b) *if $N>2$, we have $\lim_{\beta\to+\infty}\tilde s_{i,k,\beta}=\tilde c$, where $\tilde c$ is a positive constant depending only on N and $\text{meas}(\Omega)$;*
(c) *for $N\geqslant 2$, $\tilde U_{i,k,\beta}$ converges as $\beta\to+\infty$ to the radial solution U of problem (3) and the convergence is uniform on the compact subsets of $\mathbb{R}^N$.*

Notice that, also in the case of Neumann boundary conditions, there exists an infinite number of curves of the Fučík spectrum, asymptotic to the lines $\{0\}\times\mathbb{R}$ and $\mathbb{R}\times\{0\}$ and corresponding to eigenfunctions $\hat v_{k,\beta}$, which, unlike $\tilde u_{k,\beta}$, present k bumps localized near the boundary of Ω. In fact, if Ω is a smooth bounded domain of $\mathbb{R}^N$ with $N\geqslant 2$ and if Γ is a union of connected components of $\partial\Omega$, for every positive integer k there exists $\hat\beta_k>0$ having the following properties. For all $\beta>\hat\beta_k$, there exist $\hat\alpha_{k,\beta}>0$ and $\hat u_{k,\beta}\in H^1(\Omega)$, with $\hat u^+_{k,\beta}\not\equiv 0$ and $\hat u^-_{k,\beta}\not\equiv 0$, such that (4), with $\alpha=\hat\alpha_{k,\beta}$ and $u=\hat u_{k,\beta}$, is satisfied for all $\beta>\hat\beta_k$. Moreover, for all $k\in\mathbb{N}$, $\hat\alpha_{k,\beta}$ depends continuously on β, $\hat\alpha_{k,\beta}<\hat\alpha_{k+1,\beta}$ $\forall\beta>\max\{\hat\beta_k,\hat\beta_{k+1}\}$, $\hat\alpha_{k,\beta}\to 0$, as $\beta\to+\infty$, while $\hat u_{k,\beta}\to-[\text{meas}(\Omega)]^{-1/2}$ in $H^1(\Omega)$. For all $\beta>\hat\beta_k$, there exist k points $\hat x_{1,\beta},\dots,\hat x_{k,\beta}$ in Ω, satisfying $\lim_{\beta\to+\infty}\sqrt\beta\,\text{dist}(\hat x_{i,\beta},\Gamma)=0$ for $i=1,\dots,k$, such that the k bumps of $\hat u_{k,\beta}$ concentrate near those points as $\beta\to+\infty$. Moreover, if we define $\hat s_{i,k,\beta}$ and $\hat U_{i,k,\beta}$ in an analogous way as $\tilde s_{i,k,\beta}$ and $\tilde U_{i,k,\beta}$ in Theorem 2.2, then the same properties as (a) and (b) of Theorem 2.2 hold, respectively, in the cases $N=2$ and $N>2$. If (up to a subsequence) $\hat x_{i,\beta}\to\hat x_i$ as $\beta\to+\infty$, then $\hat x_i\in\Gamma\subset\partial\Omega$; if ν_i denotes the outward normal to $\partial\Omega$ in $\hat x_i$ and if we set $H_i=\{x\in\mathbb{R}^N\colon (x\cdot\nu_i)<0\}$, then we have $\lim_{\beta\to+\infty} \hat{U}_{i,k,\beta}(x) = U(x)\ \forall x \in H_i$, where U is the radial solution of problem (3); moreover, the convergence of $\hat{U}_{i,k,\beta}$ to U is uniform on the compact subsets of H_i.

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