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Weighted-composition operators on \mathcal{N}_p -spaces in the ballHu Bingyang, Khoi Le Hai¹

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ABSTRACT

In this Note, we introduce \mathcal{N}_p -spaces, some kind of Bergman-type spaces, of holomorphic functions in the unit ball of \mathbb{C}^n . Basic properties of these spaces are provided. We study weighted-composition operators between \mathcal{N}_p -spaces and the spaces A^{-q} and obtain, in particular, criteria for boundedness and compactness of such operators.

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R É S U M É

Dans cette Note, nous introduisons les espaces \mathcal{N}_p , qui sont des analogues d'espaces de Bergman de fonctions holomorphes sur la boule unité de \mathbb{C}^n . Les propriétés de base de ces espaces sont données. Nous étudions ensuite les opérateurs de composition à poids de \mathcal{N}_p dans A^{-q} , et nous obtenons, en particulier, des critères pour que ces opérateurs soient bornés ou compacts.

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1. Introduction

1.1. Notation and definitions

Let \mathbb{B} be the unit ball for the Euclidean norm in the complex vector space \mathbb{C}^n ; $\mathcal{O}(\mathbb{B})$ denotes the space of functions that are holomorphic in \mathbb{B} , with the compact-open topology, and $H^\infty(\mathbb{B})$ denotes the Banach space of bounded holomorphic functions on \mathbb{B} with the norm $\|f\|_\infty = \sup_{z \in \mathbb{B}} |f(z)|$.

If $z = (z_1, z_2, \dots, z_n)$, $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_n) \in \mathbb{C}^n$, then $\langle z, \zeta \rangle = z_1 \bar{\zeta}_1 + \dots + z_n \bar{\zeta}_n$ and $|z| = (z_1 \bar{z}_1 + \dots + z_n \bar{z}_n)^{1/2}$.

If X and Y are two topological vector spaces, then the symbol $X \hookrightarrow Y$ means the continuous embedding of X into Y .

Let $p > 0$, the Beurling-type space (sometimes also called the Bergman-type space) $A^{-p}(\mathbb{B})$ in the unit ball is defined as:

$$A^{-p}(\mathbb{B}) := \left\{ f \in \mathcal{O}(\mathbb{B}) : |f|_p = \sup_{z \in \mathbb{B}} |f(z)| (1 - |z|^2)^p < \infty \right\}.$$

For φ a holomorphic self mapping of \mathbb{B} and a holomorphic function $u : \mathbb{B} \rightarrow \mathbb{C}$, the linear operator $W_{u,\varphi} : \mathcal{O}(\mathbb{B}) \rightarrow \mathcal{O}(\mathbb{B})$:

$$W_{u,\varphi}(f)(z) = u(z) \cdot (f \circ \varphi(z)), \quad f \in \mathcal{O}(\mathbb{B}), z \in \mathbb{B},$$

is called the *weighted-composition operator* with symbols u and φ . Observe that $W_{u,\varphi}(f) = M_u C_\varphi(f)$, where $M_u(f) = uf$ is the *multiplication operator* with symbol u , and $C_\varphi(f) = f \circ \varphi$ is the *composition operator* with symbol φ . If u is identically 1, then $W_{u,\varphi} = C_\varphi$, and if φ is the identity, then $W_{u,\varphi} = M_u$.

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Composition operators and weighted composition operators acting on spaces of holomorphic functions in the unit disk \mathbb{D} of the complex plane have been studied quite well. We refer the readers to the monographs [1,6] for detailed information. Composition operators on $A^{-p}(\mathbb{D})$ have also been intensively studied (see, e.g., [2] and references therein).

1.2. \mathcal{N}_p -spaces in the unit ball

Given a point $a \in \mathbb{B}$, we can associate with it the following automorphism $\Phi_a \in \text{Aut}(\mathbb{B})$ (see, e.g., [5, pp. 25–27]):

$$\Phi_a(z) = \frac{a - P_a z - s_a Q_a z}{1 - \langle z, a \rangle},$$

where P_a is the orthogonal projection of \mathbb{C}^n on a subspace $[a]$ generated by a , that is,

$$P_a z = \begin{cases} 0, & \text{if } a = 0, \\ \frac{\langle z, a \rangle}{\langle a, a \rangle} a, & \text{if } a \neq 0, \end{cases}$$

$Q_a = I - P_a$, the projection on a orthogonal complement of $[a]$, and $s_a = (1 - |a|^2)^{1/2}$.

For $p \in (0, \infty)$, we introduce the \mathcal{N}_p -space in \mathbb{B} , which is defined as follows:

$$\mathcal{N}_p(\mathbb{B}) := \left\{ f \in \mathcal{O}(\mathbb{B}) : \|f\|_p = \sup_{a \in \mathbb{B}} \left(\int_{\mathbb{B}} |f(z)|^2 (1 - |\Phi_a(z)|^2)^p dV(z) \right)^{1/2} < \infty \right\},$$

where dV is the Lebesgue normalized volume measure on \mathbb{B} (i.e. $V(\mathbb{B}) = 1$).

It should be noted that in the case when $n = 1$, composition operators (respectively, weighted-composition operators) acting on the \mathcal{N}_p -space in the unit disk were considered in [4] (respectively, in [7]). Some results on boundedness and compactness of these operators were obtained.

The aim of the present note is to characterize the \mathcal{N}_p -spaces in the unit ball as well as the behavior of the weighted composition operators acting on these spaces. We study different properties of the weighted-composition operators and obtain the main results of [4,7] as particular cases.

2. Basic properties of \mathcal{N}_p -spaces in the ball

First we note that the automorphisms $\Phi_a(z)$ have important properties, which we list here for the reader's convenience. These properties are used in the proof of the main results of this note.

- (i) $\Phi_a(0) = a$; $\Phi_a(a) = 0$.
- (ii) $\Phi'_a(0) = -s_a^2 P_a - s_a Q_a$; $\Phi'_a(a) = -\frac{P_a}{s_a^2} - \frac{Q_a}{s_a}$.
- (iii) The identity:

$$1 - \langle \Phi_a(z), \Phi_a(w) \rangle = \frac{(1 - \langle a, a \rangle)(1 - \langle z, w \rangle)}{(1 - \langle z, a \rangle)(1 - \langle a, w \rangle)}$$

holds for any $z \in \bar{\mathbb{B}}$, $w \in \bar{\mathbb{B}}$.

- (iv) The identity:

$$1 - |\Phi_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \langle z, a \rangle|^2}$$

holds for any $z \in \bar{\mathbb{B}}$.

- (v) Φ_a is an involution: $\Phi_a(\Phi_a(z)) = z$.
- (vi) Φ_a is a homeomorphism of the closed unit ball $\bar{\mathbb{B}}$ onto itself.

It is clear that for $n = 1$, we have $P_a = I$ and $Q_a = 0$, and hence $\Phi_a(z)$ becomes the automorphism σ_a of the unit disk.

One of the most important results of this note is the following theorem, which gives various properties of \mathcal{N}_p -spaces. We note that among the statements in the theorem, assertion (a) plays a crucial role in the sequel.

Theorem 2.1. *The following statements hold:*

- (a) For $p > q > 0$, we have $H^\infty(\mathbb{B}) \hookrightarrow \mathcal{N}_q(\mathbb{B}) \hookrightarrow \mathcal{N}_p(\mathbb{B}) \hookrightarrow A^{-\frac{n+1}{2}}(\mathbb{B})$.
- (b) For $p > 0$, if $p > 2k - 1$, $k \in (0, \frac{n+1}{2}]$, then $A^{-k}(\mathbb{B}) \hookrightarrow \mathcal{N}_p(\mathbb{B})$. In particular, when $p > n$, $\mathcal{N}_p(\mathbb{B}) = A^{-\frac{n+1}{2}}(\mathbb{B})$.

(c) $\mathcal{N}_p(\mathbb{B})$ is a functional Banach space with the norm $\|\cdot\|_p$, and moreover, its norm topology is stronger than the compact-open topology.

(d) For $0 < p < \infty$, $\mathcal{B}(\mathbb{B}) \hookrightarrow \mathcal{N}_p(\mathbb{B})$, where $\mathcal{B}(\mathbb{B})$ is the Bloch space in \mathbb{B} .

Note that the results of [4, Proposition 3.1] for the unit disk are contained in assertions (a), (b) and (c), respectively, when $n = 1$.

Sketch of the proof of Theorem 2.1. (a) The first two embeddings are easy. For the third one, denote $\mathbb{B}_{1/2} = \{z: |z| < \frac{1}{2}\}$. If $f \in \mathcal{N}_p(\mathbb{B})$, then:

$$\|f\|_p^2 \geq \left(\frac{3}{4}\right)^p \int_{\mathbb{B}_{1/2}} |f(z)|^2 dV(z).$$

Furthermore, since $f \in \mathcal{O}(\mathbb{B})$, $|f|^2$ is subharmonic in \mathbb{R}^{2n} , and hence by [3, Theorem 2.1.4 (8)], we have:

$$|f(0)|^2 \leq \frac{1}{V(\mathbb{B}_{1/2})} \int_{\mathbb{B}_{1/2}} |f(z)|^2 dV(z) = 4^n \int_{\mathbb{B}_{1/2}} |f(z)|^2 dV(z). \tag{2.1}$$

Thus,

$$|f(0)|^2 \leq \frac{4^{p+n}}{3^p} \|f\|_p^2, \quad f \in \mathcal{N}_p(\mathbb{B}). \tag{2.2}$$

For every fixed $z \in \mathbb{B}$, we put

$$F_{z,f}(w) = (f \circ \Phi_z(w)) \cdot \frac{(1 - |z|^2)^{\frac{n+1}{2}}}{(1 - \langle w, z \rangle)^{n+1}}, \quad w \in \mathbb{B},$$

which is clearly a holomorphic function in \mathbb{B} . We can prove that $\|F_{z,f}\|_p^2 \leq \|f\|_p^2$, and so $F_{z,f} \in \mathcal{N}_p(\mathbb{B})$. Then, by (2.2), we have:

$$|f(z)|^2 (1 - |z|^2)^{n+1} = |F_{z,f}(0)|^2 \leq \frac{4^{p+n}}{3^p} \|F_{z,f}\|_p^2 \leq \frac{4^{p+n}}{3^p} \|f\|_p^2, \quad \forall z \in \mathbb{B},$$

which implies that:

$$|f|_{\frac{n+1}{2}} = \sup_{z \in \mathbb{B}} |f(z)| (1 - |z|^2)^{\frac{n+1}{2}} \leq \frac{2^{p+n}}{3^{p/2}} \|f\|_p, \quad \forall f \in \mathcal{N}_p(\mathbb{B}). \tag{2.3}$$

That is, $\mathcal{N}_p(\mathbb{B}) \hookrightarrow A^{-\frac{n+1}{2}}(\mathbb{B})$.

(b) Suppose $p > \max\{0, 2k - 1\}$, where $k \in (0, \frac{n+1}{2}]$. Let $f \in A^{-k}(\mathbb{B})$, we have:

$$\|f\|_p^2 \leq |f|_k^2 \sup_{a \in \mathbb{B}} (1 - |a|^2)^p \int_{\mathbb{B}} \frac{(1 - |z|^2)^{p-2k}}{|1 - \langle z, a \rangle|^{2p}} dV(z).$$

Moreover, by [5, Proposition 1.4.10], we have for some positive constant C :

$$(1 - |a|^2)^p \int_{\mathbb{B}} \frac{(1 - |z|^2)^{p-2k}}{|1 - \langle z, a \rangle|^{2p}} dV(z) \leq C, \quad \forall a \in \mathbb{B},$$

which implies that

$$\|f\|_p \leq \sqrt{C} |f|_k, \quad \forall f \in \mathcal{N}_p(\mathbb{B}).$$

That is $A^{-k}(\mathbb{B}) \hookrightarrow \mathcal{N}_p(\mathbb{B})$.

(c) and (d) are easy. \square

3. Weighted composition operators between $\mathcal{N}_p(\mathbb{B})$ and $A^{-q}(\mathbb{B})$

The following “probe” functions in \mathcal{N}_p -spaces are important in the proof of our main results in this section.

Lemma 3.1. For each $w \in \mathbb{B}$, put:

$$k_w(z) := \left(\frac{1 - |w|^2}{(1 - \langle z, w \rangle)^2} \right)^{\frac{n+1}{2}}, \quad z \in \mathbb{B}.$$

Then $k_w \in \mathcal{N}_p(\mathbb{B})$ and $\sup_{w \in \mathbb{B}} \|k_w\|_p \leq 1$.

3.1. Boundedness

Note that the norm topology of both $\mathcal{N}_p(\mathbb{B})$ and $A^{-q}(\mathbb{B})$ is stronger than the compact-open topology, and hence it is stronger than the pointwise convergence topology. Thus, if the weighted-composition operator $W_{u,\varphi}$ maps $\mathcal{N}_p(\mathbb{B})$ into $A^{-q}(\mathbb{B})$, an application of closed graph theorem shows that $W_{u,\varphi}$ is automatically bounded from $\mathcal{N}_p(\mathbb{B})$ into $A^{-q}(\mathbb{B})$.

Theorem 3.2. Let $\varphi: \mathbb{B} \rightarrow \mathbb{B}$ be a holomorphic mapping, $u: \mathbb{B} \rightarrow \mathbb{C}$ a holomorphic mapping and $p, q > 0$. The weighted composition operator $W_{u,\varphi}: \mathcal{N}_p(\mathbb{B}) \rightarrow A^{-q}(\mathbb{B})$ is bounded if and only if:

$$\sup_{z \in \mathbb{B}} |u(z)| \frac{(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{\frac{n+1}{2}}} < \infty. \quad (3.1)$$

Note when $n = 1$, [Theorem 3.2](#) contains Theorem 3 in [\[7\]](#) as a particular case.

Sketch of the proof of Theorem 3.2. If $W_{u,\varphi}: \mathcal{N}_p(\mathbb{B}) \rightarrow A^{-q}(\mathbb{B})$ is bounded, then by [Lemma 3.1](#), there exists a positive constant M such that:

$$M \geq |W_{u,\varphi}(k_{\varphi(z)})|_q \geq |u(z)| \frac{(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{\frac{n+1}{2}}}, \quad \forall z \in \mathbb{B},$$

from which [\(3.1\)](#) follows.

Conversely, from [\(3.1\)](#), by [Theorem 2.1\(a\)](#), it follows that for some positive constant M :

$$|W_{u,\varphi}(f)|_q \leq \sup_{z \in \mathbb{B}} |u(z)| \frac{(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{\frac{n+1}{2}}} \cdot |f|_{\frac{n+1}{2}} \leq M \|f\|_p,$$

which shows that $W_{u,\varphi}$ is bounded from $\mathcal{N}_p(\mathbb{B})$ into $A^{-q}(\mathbb{B})$. \square

3.2. Compactness

By analogous arguments as of [\[1, Proposition 3.11\]](#), we can obtain the following test for compactness of $W_{u,\varphi}$.

Lemma 3.3. Let $\varphi: \mathbb{B} \rightarrow \mathbb{B}$ be a holomorphic mapping, $u: \mathbb{B} \rightarrow \mathbb{C}$ a holomorphic mapping and $p, q > 0$. The weighted composition operator $W_{u,\varphi}: \mathcal{N}_p(\mathbb{B}) \rightarrow A^{-q}(\mathbb{B})$ is compact if and only if $|W_{u,\varphi}(f_m)|_q \rightarrow 0$, as $m \rightarrow \infty$ for any bounded sequence $\{f_m\}$ in $\mathcal{N}_p(\mathbb{B})$ which converges to 0 uniformly on every compact subset of \mathbb{B} .

We have the following result.

Theorem 3.4. Let $\varphi: \mathbb{B} \rightarrow \mathbb{B}$ be a holomorphic mapping, $u: \mathbb{B} \rightarrow \mathbb{C}$ a holomorphic mapping and $p, q > 0$. The weighted composition operator $W_{u,\varphi}: \mathcal{N}_p(\mathbb{B}) \rightarrow A^{-q}(\mathbb{B})$ is compact if and only if:

$$\lim_{r \rightarrow 1^-} \sup_{|\varphi(z)| > r} |u(z)| \frac{(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{\frac{n+1}{2}}} = 0. \quad (3.2)$$

When $n = 1$, [Theorem 3.4](#) contains Corollary 2 in [\[7\]](#) as a particular case.

Sketch of the proof of Theorem 3.4. If $W_{u,\varphi}: \mathcal{N}_p(\mathbb{B}) \rightarrow A^{-q}(\mathbb{B})$ is compact, then it is bounded. By [Theorem 3.2](#), $M = \sup_{z \in \mathbb{B}} |u(z)| \frac{(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{\frac{n+1}{2}}} < \infty$.

Note that $\lim_{r \rightarrow 1^-} F(r)$, where:

$$F(r) = \sup_{|\varphi(z)| > r} |u(z)| \frac{(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{\frac{n+1}{2}}},$$

always exists. We show that (3.2) holds.

Assume on the contrary that $\lim_{r \rightarrow 1^-} F(r) = L > 0$. There exists an $r_0 \in (0, 1)$ such that for all $r \in (r_0, 1)$, we have $F(r) > L/2$. Then by the standard diagonal process, we can construct a sequence $\{z_m\} \subset \mathbb{B}$ such that $|\varphi(z_m)| \rightarrow 1$ as $m \rightarrow \infty$, and also for each $m \in \mathbb{N}$, $|u(z_m)| \cdot \frac{(1 - |z_m|^2)^q}{(1 - |\varphi(z_m)|^2)^{\frac{n+1}{2}}} \geq L/4$. Consider the probe functions k_{w_m} , where $w_m = \varphi(z_m)$, defined in Lemma 3.1. It is easy to see that $k_{w_m} \rightarrow 0$ uniformly on every compact subset of \mathbb{B} . Moreover, for each $m \in \mathbb{N}$, $\|k_{w_m}\|_p \leq 1$. Since $W_{u,\varphi}$ is compact, by Lemma 3.3, $|W_{u,\varphi}(k_{w_m})|_q \rightarrow 0$ as $m \rightarrow \infty$. However, for each $m \in \mathbb{N}$,

$$|W_{u,\varphi}(k_{w_m})|_q \geq |u(z_m)| \cdot |k_{w_m}(\varphi(z_m))| (1 - |z_m|^2)^q = |u(z_m)| \cdot \frac{(1 - |z_m|^2)^q}{(1 - |\varphi(z_m)|^2)^{\frac{n+1}{2}}} \geq \frac{L}{4},$$

which gives a contradiction.

Conversely, if (3.2) holds, and $\{f_m\}$ is a bounded sequence in $\mathcal{N}_p(\mathbb{B})$ which converges to zero uniformly on every compact subset of \mathbb{B} , then for a given $\varepsilon > 0$, there exist $r_0 \in (0, 1)$ and $m_\varepsilon \in \mathbb{N}$, such that for $r \in (r_0, 1)$ and $m > m_\varepsilon$, we have:

$$|W_{u,\varphi}(f_m)|_q \leq \sup_{|\varphi(z)| > r} |u(z)| |f_m(\varphi(z))| (1 - |z|^2)^q + \sup_{|\varphi(z)| \leq r} |u(z)| |f_m(\varphi(z))| (1 - |z|^2)^q < \varepsilon.$$

From this it follows that $|W_{u,\varphi}(f_m)|_q \rightarrow 0$ as $m \rightarrow \infty$. By Lemma 3.3, we get the desired result. \square

As a corollary of Theorems 3.2 and 3.4, we have:

Corollary 3.5. Let $\varphi: \mathbb{B} \rightarrow \mathbb{B}$ be a holomorphic mapping and $p, q > 0$. The composition operator C_φ acting from $\mathcal{N}_p(\mathbb{B}) \rightarrow A^{-q}(\mathbb{B})$

(1) is bounded if and only if

$$\sup_{z \in \mathbb{B}} \frac{(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{\frac{n+1}{2}}} < \infty,$$

(2) is compact if and only if

$$\lim_{r \rightarrow 1^-} \sup_{|\varphi(z)| > r} \frac{(1 - |z|^2)^q}{(1 - |\varphi(z)|^2)^{\frac{n+1}{2}}} = 0.$$

When $n = 1$, Corollary 3.5 contains Theorems 4.1 and 4.3 in [4] as particular cases.

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