C. R. Acad. Sci. Paris, Ser. I 351 (2013) 833–835

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Differential geometry

A note on Chow's entropy functional for the Gauss curvature flow

Note sur la fonctionnelle d'entropie de Chow relative au flot de la courbure de Gauss

Hongxin Guo [a], Robert Philipowski [b], Anton Thalmaier [b]

[a] *School of Mathematics and Information Science, Wenzhou University, Wenzhou, Zhejiang 325035, China*
[b] *Unité de Recherche en Mathématiques, FSTC, Université du Luxembourg, 6, rue Richard-Coudenhove-Kalergi, L-1359 Luxembourg, Luxembourg*

ARTICLE INFO

Article history:
Received 19 September 2013
Accepted after revision 7 October 2013
Available online 28 October 2013

Presented by the Editorial Board

ABSTRACT

Based on the entropy formula for the Gauss curvature flow introduced by Bennett Chow, we define an entropy functional that is monotone along the unnormalized flow and whose critical point is a shrinking self-similar solution.

RÉSUMÉ

À partir de la formule d'entropie introduite par Bennett Chow pour le flot de la courbure de Gauss, nous définissons une entropie qui est monotone le long du flot non normalisé, et dont le point critique est une solution auto-similaire contractante.

1. Introduction and the main theorem

Entropy formulas are powerful tools to study geometric flows. One famous example is Perelman's $\mathcal{W}$ entropy functional for the Ricci flow [6]. Prior to Perelman's entropy, Hamilton had applied an entropy formula to study Ricci flow on closed surfaces with positive curvature [4]. While there seemed to be no obvious relationship between these two entropy formulas, recently the first author was able to define a new entropy formula relating them [3].

For the Gauss curvature flow, similar to Hamilton's surface entropy, there is an entropy functional studied by Chow [1] and Hamilton [5].

Let M be a hypersurface parameterized by a map $F: M^n \to \mathbb{R}^{n+1}$. The Gauss curvature flow of M is given by:

$$\begin{cases} \dfrac{\partial F(x,t)}{\partial t} = -K(x,t)\nu(x,t) \\ F(x,0) = F_0(x) \end{cases} \tag{1.1}$$

where K denotes the Gaussian curvature, ν denotes the outward unit normal vector field, and F_0 parameterizes the initial surface.

Throughout this paper, we assume that $F_0(M)$ is convex, and thus $F(M,t)$ remains convex along (1.1). The volume normalized Gauss curvature flow is:

E-mail addresses: guo@wzu.edu.cn (H. Guo), robert.philipowski@uni.lu (R. Philipowski), anton.thalmaier@uni.lu (A. Thalmaier).

1631-073X/$ – see front matter
http://dx.doi.org/10.1016/j.crma.2013.10.003

$$\frac{\partial F(x,t)}{\partial t} = -\big(K(x,t) - u(x,t)\big)\nu(x,t) \tag{1.2}$$

where $u(x,t) = \langle F(x,t), \nu(x,t)\rangle$ is the supporting function.

Chow studied the following entropy functional

$$\mathcal{N} = \int_M K \log K \, d\mu \tag{1.3}$$

and proved that (1.3) is monotone along the normalized Gauss curvature flow (1.2). Recently, a new entropy was defined by Guan and Ni [2]; their entropy is monotone along the normalized flow.

On the other hand, for the unnormalized flow (1.1) which shrinks the convex hypersurface to a point, the model case is a shrinking self-similar solution.

In view of Perelman's $\mathcal{W}$ entropy, which is monotone along the unnormalized Ricci flow and whose critical point is a shrinking soliton, we are interested in searching for an entropy that is monotone along the unnormalized flow and whose critical point is a shrinking self-similar solution. In this paper, we define such an entropy.

We use the same notation as in [1]. Let:

$$P_{ij} = \nabla_i \nabla_j K - h^{-1}_{k\ell} \nabla_k h_{ij} \nabla_\ell K + K h^2_{ij}$$

and P be its trace with respect to the second fundamental form, namely $P = h^{-1}_{ij} P_{ij}$.

The main result of this short note is the following theorem.

Theorem 1.1. *Let M^n be a compact evolving hypersurface of $\mathbb{R}^{n+1}$. Suppose that the Gauss curvature satisfies $K > 0$ and let $\tau > 0$. We define*:

$$\mathcal{W} \doteqdot \int_M \big(\tau P - (n+1)\log K - n \log \tau\big) K \, d\mu. \tag{1.4}$$

If τ satisfies:

$$\frac{d\tau}{dt} = -(n+1) \tag{1.5}$$

and M is a convex hypersurface evolving according to the Gauss curvature flow (1.1)*, then*

$$\frac{d\mathcal{W}}{dt} = \tau \int_M \left(\left| P_{ij} - \frac{1}{\tau} h_{ij} \right|_h^2 + \left(P - \frac{n}{\tau} \right)^2 \right) K \, d\mu. \tag{1.6}$$

In particular, $\mathcal{W}$ is monotone increasing along (1.1)*, and the monotonicity is strict unless the solution satisfies*:

$$P_{ij} - \frac{1}{\tau} h_{ij} = 0. \tag{1.7}$$

2. An example and proof of the theorem

2.1. Entropy on a canonical shrinking sphere

Before proving the main theorem, we compute the entropy $\mathcal{W}$ on a shrinking sphere as a solution to the Gauss curvature flow.

Suppose that M^n is a shrinking sphere of radius $r(t)$. Then the Riemannian metric is given by $g(t) = r^2(t) g_0$ where g_0 is the standard sphere of radius 1. The second fundamental form is given by $h(t) = r(t) g_0 = g(t)/r(t)$, and the Gauss curvature by $K(t) = r^{-n}$. Thus, when M^n evolves along the Gauss curvature flow (1.1), we have:

$$\frac{dr(t)}{dt} = -\frac{1}{r^n(t)}$$

and $r(t) = (1 - (n+1)t)^{1/(n+1)} = \tau^{1/(n+1)}$. Furthermore, we have $K = \tau^{-n/(n+1)}$. By definition, we get $P_{ij} = \tau^{-1} h_{ij}$ and $P = n/\tau$. Thus, on a canonical shrinking sphere solution to the Gaussian curvature flow, the entropy is given by:

$$\mathcal{W} = n\sigma_n$$

where σ_n denotes the volume of the sphere of radius 1.

2.2. Proof of Theorem 1.1

Recall that in [1] it has been shown that:

$$\frac{\mathrm{d}\mathcal{N}}{\mathrm{d}t}=\int_M PK\,\mathrm{d}\mu$$

and

$$\frac{\mathrm{d}^2\mathcal{N}}{\mathrm{d}t^2}=\int_M \left(|P_{ij}|_h^2+P^2\right)K\,\mathrm{d}\mu. \tag{2.8}$$

Starting from (2.8), we have:

$$\begin{aligned}\frac{\mathrm{d}^2\mathcal{N}}{\mathrm{d}t^2}&=\int_M \left(|P_{ij}|_h^2+P^2\right)K\,\mathrm{d}\mu\\
&=\int_M\left(\left|P_{ij}-\frac{1}{\tau}h_{ij}\right|_h^2+\frac{2}{\tau}\langle h_{ij},P_{ij}\rangle_h-\frac{1}{\tau^2}|h_{ij}|_h^2\right)K\,\mathrm{d}\mu+\int_M\left(\left(P-\frac{n}{\tau}\right)^2+\frac{2n}{\tau}P-\frac{n^2}{\tau^2}\right)K\,\mathrm{d}\mu\\
&=\int_M\left(\left|P_{ij}-\frac{1}{\tau}h_{ij}\right|_h^2+\frac{2}{\tau}P-\frac{n}{\tau^2}\right)K\,\mathrm{d}\mu+\int_M\left(\left(P-\frac{n}{\tau}\right)^2+\frac{2n}{\tau}P-\frac{n^2}{\tau^2}\right)K\,\mathrm{d}\mu\\
&=\int_M\left(\left|P_{ij}-\frac{1}{\tau}h_{ij}\right|_h^2+\left(P-\frac{n}{\tau}\right)^2\right)K\,\mathrm{d}\mu+\frac{2(n+1)}{\tau}\int_M PK\,\mathrm{d}\mu-\frac{n+n^2}{\tau^2}\int_M K\,\mathrm{d}\mu\end{aligned}$$

and this can be rewritten as:

$$\int_M\left(\left|P_{ij}-\frac{1}{\tau}h_{ij}\right|_h^2+\left(P-\frac{n}{\tau}\right)^2\right)K\,\mathrm{d}\mu=\frac{\mathrm{d}^2\mathcal{N}}{\mathrm{d}t^2}-\frac{2(n+1)}{\tau}\frac{\mathrm{d}\mathcal{N}}{\mathrm{d}t}+\frac{n(n+1)}{\tau^2}\int_M K\,\mathrm{d}\mu.$$

Noting that $\int_M K\,\mathrm{d}\mu$ stays constant along the Gauss curvature flow, and using $\mathrm{d}\tau/\mathrm{d}t=-(n+1)$, we rewrite the right-hand side of the above equation as:

$$\frac{\mathrm{d}^2\mathcal{N}}{\mathrm{d}t^2}-\frac{2(n+1)}{\tau}\frac{\mathrm{d}\mathcal{N}}{\mathrm{d}t}+\frac{n(n+1)}{\tau^2}\int_M K\,\mathrm{d}\mu=\frac{1}{\tau}\frac{\mathrm{d}}{\mathrm{d}t}\left(\tau\frac{\mathrm{d}\mathcal{N}}{\mathrm{d}t}-(n+1)\mathcal{N}-n\log\tau\int_M K\,\mathrm{d}\mu\right)$$

and moreover we arrive at:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\tau\frac{\mathrm{d}\mathcal{N}}{\mathrm{d}t}-(n+1)\mathcal{N}-n\log\tau\int_M K\,\mathrm{d}\mu\right)=\tau\int_M\left(\left|P_{ij}-\frac{1}{\tau}h_{ij}\right|_h^2+\left(P-\frac{n}{\tau}\right)^2\right)K\,\mathrm{d}\mu.$$

The above computations suggest to define:

$$\mathcal{W}=\tau\frac{\mathrm{d}\mathcal{N}}{\mathrm{d}t}-(n+1)\mathcal{N}-n\log\tau\int_M K\,\mathrm{d}\mu$$

as in (1.4). Then (1.6) holds and this completes the proof of the theorem.

Acknowledgements

Research supported by NSFC (Grants No. 11001203 and 11171143), Zhejiang Provincial Natural Science Foundation of China (Project No. LY13A010009) and Fonds national de la recherche, Luxembourg.

References

[1] Bennett Chow, On Harnack's inequality and entropy for the Gaussian curvature flow, Commun. Pure Appl. Math. 4 (4) (1991) 469–483.
[2] Pengfei Guan, Lei Ni, Entropy and a convergence theorem for Gauss curvature flow in high dimension, arXiv:1306.0625 [math.DG].
[3] Hongxin Guo, An entropy formula relating Hamilton's surface entropy and Perelman's $\mathcal{W}$ entropy, C. R. Acad. Sci. Paris, Ser. I 351 (3–4) (2013) 115–118.
[4] Richard S. Hamilton, The Ricci flow on surfaces, in: Mathematics and General Relativity, Santa Cruz, CA, 1986, in: Contemp. Math., vol. 71, 1988, pp. 237–262.
[5] Richard S. Hamilton, Remarks on the entropy and Harnack estimates for the Gauss curvature flow, Commun. Anal. Geom. 2 (1) (1994) 155–165.
[6] Grisha Perelman, The entropy formula for the Ricci flow and its geometric applications, arXiv:math/0211159v1.