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Differential topology

The displaced disks problem via symplectic topology

Le problème des disques déplacés via la topologie symplectique

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ABSTRACT

We prove that a C^0-small area preserving the homeomorphism of a closed surface with vanishing mass flow cannot displace a topological disk of large area. This resolves the displaced disks problem posed by F. Béguin, S. Crovisier, and F. Le Roux.

RÉSUMÉ

Nous démontrons qu'une petite surface C^0 préservant l'homéomorphisme d'une surface fermée avec un flux de masse disparaissant ne peut pas déplacer un disque topologique de grande surface. Ceci résout le problème des disques déplacés posé par F. Béguin, S. Crovisier et F. Le Roux.

1. Introduction

Let Σ be a closed surface endowed with an area form Ω. We denote by $Homeo^{\Omega}(\Sigma)$ the group of area preserving the homeomorphisms of Σ, by $Homeo_0^{\Omega}(\Sigma)$ the path component of identity in $Homeo^{\Omega}(\Sigma)$, and by $Ham(\Sigma)$ the group of Hamiltonian diffeomorphisms of Σ. We will be studying, $\mathcal{G}$, the C^0 closure of $Ham(\Sigma)$ inside $Homeo_0^{\Omega}(\Sigma)$.

The group $\mathcal{G}$ is a well-studied dynamical object: it is precisely the set of elements of $Homeo_0^{\Omega}(\Sigma)$ with vanishing *mass flow*. For a definition of the mass flow homomorphism, which is also known as the *mean rotation vector*, see Section 5 of [3]. Equivalently, $\mathcal{G}$ can be described as the set of elements of $Homeo_0^{\Omega}(\Sigma)$ with zero flux; see Appendix A.5 of [3]. In this note, we only work with the description of $\mathcal{G}$ as the C^0 closure of $Ham(\Sigma)$. It is well known that in the case of S^2, $\mathcal{G} = Homeo_0^{\Omega}(S^2)$.

Recall that a homeomorphism ϕ is said to displace a set B if $\phi(B) \cap B = \emptyset$. For $a > 0$, define $\mathcal{G}_a = \{\theta \in \mathcal{G}\colon\ \theta \text{ displaces a topological disk of area at least } a\}$. The displaced disks problem, posed by F. Béguin, S. Crovisier, and F. Le Roux, asks the following.

Question. (See Béguin, Crovisier, Le Roux [1].) Does the C^0 closure of $\mathcal{G}_a$ contain the identity?

The initial motivation of Béguin, Crovisier, and Le Roux for posing this beautiful question is as follows: $\mathcal{G}$ is a normal subgroup of $Homeo^{\Omega}(\Sigma)$. Béguin, Crovisier, and Le Roux were interested in knowing whether the conjugacy class of an

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element of $\mathcal{G}$ could be C^0-dense in all of $\mathcal{G}$. As we will see below, a negative answer to the displaced disks problem provides a negative answer to this question as well.

Equip Σ with a Riemannian distance d and define the C^0 distance between two homeomorphisms ϕ, ψ by $d_{C^0}(\phi,\psi) := \max\{\max_{x\in\Sigma} d(\phi(x),\psi(x)), \max_{x\in\Sigma} d(\phi^{-1}(x),\psi^{-1}(x))\}$. Our main result provides a negative answer to the displaced disks problem.

Theorem 1. *There exists $\epsilon > 0$ such that $d_{C^0}(Id,\theta) \geqslant \epsilon$ for all $\theta \in \mathcal{G}_a$.*

Let $C(\phi) = \{\psi\phi\psi^{-1}\colon\ \psi \in Homeo^{\Omega}(\Sigma)\}$ denote the conjugacy class of $\phi \in \mathcal{G}$. Observe that $\mathcal{G}_a$ is invariant under a conjugation by area preserving homeomorphisms and hence, if $\phi \neq Id$, then $C(\phi) \subset \mathcal{G}_a$, for some $a > 0$. Theorem 1 immediately yields the next result.

Corollary 2. *$C(\phi)$ is not C^0-dense in $\mathcal{G}$.*

Remark. Working independently of the author, Andrew D. Hanlon and Daniel N. Dore [6] have obtained results similar to what appears in this article.

The following facts were pointed out to me by Béguin, Crovisier, and Le Roux.

1. Theorem 1 would not hold without requiring $\theta \in \mathcal{G}$. Indeed, it is not difficult to see that a C^0-small translation of the torus displaces a disk with an area nearly equal to half the total area of the torus.

2. Corollary 2 does not hold for arbitrary elements of $Homeo_0(\Sigma)$. See Remark 7.11 of [5] for an example of a homeomorphism of S^2 whose conjugacy class is C^0-dense in $Homeo_0(S^2)$. Once it is established that the conjugacy class of a homeomorphism is dense, then it is easy to see that the conjugacy class of that homeomorphism is a G_δ set. Hence, we conclude that the conjugacy class of a generic homeomorphism of S^2 is dense.

3. Suppose that $\Sigma \neq S^2$. It follows from the work of Gaumbado and Ghys [4] and Entov, Polterovich, and Py [2] that $\mathcal{G}$ carries C^0-continuous and homogeneous quasimorphisms; see Theorem 1.2 of [2]. Corollary 2 follows immediately as homogeneous quasimorphisms are constant on conjugacy classes.

2. Proof of Theorem 1

Our proof uses Floer theoretic invariants of Hamiltonian diffeomorphisms. In particular, we use the theory of spectral invariants, or action selectors, introduced by C. Viterbo, M. Schwarz, and Y.-G. Oh [10,8,7]. An important consequence of this theory is that the group of Hamiltonian diffeomorphisms of a closed symplectic manifold M admits a conjugation invariant norm $\gamma : Ham(M) \to [0,\infty)$.[1] Being a conjugation invariant norm, γ satisfies the following axioms:

(i) $\gamma(\phi) \geqslant 0$ with equality if and only if $\phi = Id$,
(ii) $\gamma(\phi) = \gamma(\phi^{-1})$,
(iii) $\gamma(\phi\psi) \leqslant \gamma(\phi) + \gamma(\psi)$,
(iv) $\gamma(\psi\phi\psi^{-1}) = \gamma(\phi)$.

An important feature of γ is the fact that it satisfies the so-called *energy-capacity* inequality. In the case of a closed surface Σ the energy-capacity inequality states that if $\phi \in Ham(\Sigma)$ displaces a disk of area a, then:

$$a \leqslant \gamma(\phi). \tag{1}$$

Theorem 2 of [9] provides the final step of our solution. According to this theorem, for a closed surface Σ, of genus g, there exist constants $C,\delta > 0$ such that $\forall\phi \in Ham(\Sigma)$ if $d_{C^0}(Id,\phi) \leqslant \delta$, then

$$\gamma(\phi) \leqslant C\, d_{C^0}(Id,\phi)^{2^{-2g-1}}. \tag{2}$$

We now prove Theorem 1. For a contradiction, suppose it does not hold and pick a sequence $\theta_i \in \mathcal{G}_a$ that converges uniformly to the identity and conclude from Inequality (2) that $\gamma(\theta_i) \to 0$. But this is impossible because the energy-capacity inequality (1) implies that $\gamma|_{\mathcal{G}_a} \geqslant a$.

2.1. Extending γ to $\mathcal{G}$

We will finish this note by showing that the conjugation invariant norm γ extends continuously to $\mathcal{G}$. We need a small modification of Inequality (2). Let C, δ denote the constants appearing in this inequality and suppose that $d_{C^0}(\psi,\phi) \leqslant \delta$, where $\psi,\phi \in Ham(\Sigma)$. Applying Inequality (2) to $\psi\phi^{-1}$, we obtain that $\gamma(\psi\phi^{-1}) \leqslant C\, d_{C^0}(Id,\psi\phi^{-1})^{2^{-2g-1}} \leqslant$

[1] γ is usually defined on the universal cover of Ham. For $\phi \in Ham$, one can define $\gamma(\phi)$ by taking infimum over all paths which end at ϕ.

$C\,d_{C^0}(\phi,\psi)^{2^{-2g-1}}$; the latter inequality follows from the definition of d_{C^0}. Now, using Axiom (iii) of γ, we see that $\gamma(\psi) - \gamma(\phi) \leqslant \gamma(\psi\phi^{-1})$. Hence, $\gamma(\psi) - \gamma(\phi) \leqslant C\,d_{C^0}(\psi,\phi)^{2^{-2g-1}}$. Similarly, we obtain the same upper bound for $\gamma(\phi) - \gamma(\psi)$. Therefore, we have proven that $\forall \psi, \phi \in Ham(\Sigma)$ if $d_{C^0}(\psi,\phi) \leqslant \delta$, then

$$\left|\gamma(\psi) - \gamma(\phi)\right| \leqslant C\,d_{C^0}(\psi,\phi)^{2^{-2g-1}}.$$

We see that γ is uniformly continuous with respect to d_{C^0} and so, it extends continuously to $\mathcal{G}$. Clearly, the extension $\gamma : \mathcal{G} \to \mathbb{R}$ satisfies Inequalities (1) and (2), in addition to the four stated axioms of conjugation invariant norms.

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