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Partial differential equations

Cauchy problem for effectively hyperbolic operators with triple characteristics

Problème de Cauchy pour des opérateurs effectivement hyperboliques ayant des caractéristiques triples

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A R T I C L E   I N F O

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A B S T R A C T

We study a class of third-order effectively hyperbolic operators P in $G = \{(t, x): 0 \leqslant t \leqslant T, x \in U \Subset \mathbb{R}^n\}$ with triple characteristics at $\rho = (0, x_0, \xi), \xi \in \mathbb{R}^n \setminus \{0\}$. V. Ivrii introduced the conjecture that every effectively hyperbolic operator is *strongly hyperbolic*, that is the Cauchy problem for $P + Q$ is locally well posed for any lower-order terms Q. For operators with triple characteristics, this conjecture was established [3] in the case when the principal symbol of P admits a factorization as a product of two symbols of principal type. A strongly hyperbolic operator in G could have triple characteristics in G only for $t = 0$ or for $t = T$. The operators that we investigate have a principal symbol which in general is not factorizable and we prove that these operators are strongly hyperbolic if T is small enough.

R É S U M É

On étudie une classe d'opérateurs effectivement hyperboliques P dans $G = \{(t, x): 0 \leqslant t \leqslant T, x \in U \Subset \mathbb{R}^n\}$ ayant des caractéristiques triples pour $\rho = (0, x_0, \xi), \xi \in \mathbb{R}^n \setminus \{0\}$. V. Ivrii a introduit la conjecture selon laquelle chaque opérateur effectivement hyperbolique est fortement hyperbolique, c'est-à-dire telle que le problème de Cauchy pour $P + Q$ soit localement bien posé pour tout opérateur Q d'ordre inférieur à celui de P. Pour des opérateurs ayant des caractéristiques triples, cette conjecture a été démontrée [3] pour le cas où le symbole principal de P admet une factorisation comme produit de deux symboles du type principal. Un opérateur fortement hyperbolique pourrait avoir des caractéristiques triples seulement pour $t = 0$ ou pour $t = T$. Les opérateurs que nous examinons ont en général un symbole principal qui n'est pas factorisable, et nous prouvons qu'ils sont fortement hyperboliques si T est suffisamment petit.

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1. Introduction and main result

Consider a differential operator $P(t,x,D_t,D_x)=\sum_{\alpha+|\beta|\leqslant m} c_{\alpha,\beta}(t,x)D_t^{\alpha}D_x^{\beta}$, $D_t=-\mathbf{i}\partial_t$, $D_{x_j}=-\mathbf{i}\partial_{x_j}$ of order m with C^∞ coefficients $c_{\alpha,\beta}(t,x)$, $t\in\mathbb{R}$, $x\in\mathbb{R}^n$. Denote by $p_m(t,x,\tau,\xi)$ the principal symbol of P. We assume that $c_{m,0}(t,x)=1$ for all (t,x). Let $\Omega\subset\mathbb{R}^{n+1}$ be an open set and let $\Omega_\eta^-=\Omega\cap\{t\leqslant\eta\}$, $\Omega_\eta^+=\Omega\cap\{t\geqslant\eta\}$, $G=\Omega\cap\{0\leqslant t\leqslant T\}$. Set $P_m(t,x,D_t,D_x)=p_m(t,x,D_t,D_x)$.

Definition 1. We say that the Cauchy problem:

$$Pu=f \quad \text{in } \Omega\cap\{t<T\}, \qquad \operatorname{supp} u\subset\overline{G} \tag{1.1}$$

is well posed in G if: (i) (existence) for every $f\in C_0^\infty(\Omega)$, $\operatorname{supp} f\subset\overline{\Omega_0^+}$ there exists a solution $u\in\mathcal{D}'(\Omega)$ satisfying (1.1), (ii) (uniqueness) if $u\in\mathcal{D}'(\Omega)$ satisfies (1.1), then for every s, $0<s\leqslant T$, if $Pu=0$ in Ω_s^-, then $u=0$ in Ω_s^-.

A necessary condition for the well-posedness of the Cauchy problem (WPC) is the hyperbolicity of the operator P in G (see [4] and the references cited there). This means that for every $(t_0,x_0,\xi)\in G\times\mathbb{R}^n\setminus\{0\}$, the equation $p_m(t_0,x_0,\tau,\xi)=0$ with respect to τ has only real roots $\tau=\lambda_j(t_0,x_0,\xi)$.

Definition 2. We say that the operator P with principal symbol p_m is strongly hyperbolic in G if for every point $z_0=(t_0,x_0)\in G$ there exists a neighborhood U of z_0, $T(U)>0$ and $T_0\geqslant 0$ ($T_0<T$ if $t_0=T$ and $T_0=0$ if $t_0=0$) such that the Cauchy problem (1.1) for the operator $L=P_m(t,x,D_t,D_x)+Q_{m-1}(t,x,D_t,D_x)$ is well posed in $U\cap\{s\leqslant t\leqslant T(U)\}$ for every $T_0\leqslant s<T(U)$ and for any operator $Q_{m-1}(t,x,D_t,D_x)$ of order less or equal to $m-1$.

In the following, we switch to a different notation for the sake of simplicity and denote $t=x_0$, $x=(x_0,x_1,\dots,x_n)\in\mathbb{R}^{n+1}$. The dual variables are denoted by $\xi=(\xi_0,\xi_1,\dots,\xi_n)=(\xi_0,\xi')$. Given a symbol $p(x,\xi)$, let $\Sigma(p)=\{z\in T^*G\setminus\{0\}$: $p(z)=0\}$, $\Sigma_1(p)=\{z\in T^*(G)\setminus\{0\}$: $p(z)=0, dp(z)=0\}$. In the case $\Sigma_1(p_m)=\emptyset$, the operator is of principal type and a hyperbolic operator P in G is strongly hyperbolic (see Section 23.4 in [2] and [3]). Passing to the case $\Sigma_1(p_m)\neq\emptyset$, for $(\hat{x},\hat{\xi})\in\Sigma_1(p_m)$, we are led to define the fundamental matrix:

$$F_p(\hat{x},\hat{\xi})=\begin{pmatrix} p_{\xi,x}(\hat{x},\hat{\xi}) & p_{\xi,\xi}(\hat{x},\hat{\xi}) \\ -p_{x,x}(\hat{x},\hat{\xi}) & -p_{x,\xi}(\hat{x},\hat{\xi}) \end{pmatrix}.$$

If P is hyperbolic in G and $(\hat{x},\hat{\xi})$ is a critical point of $p_m(x,\xi)$, then $F_{p_m}(\hat{x},\hat{\xi})$ has at most two non-vanishing real simple eigenvalues μ and $-\mu$ and all other eigenvalues μ_j are purely imaginary, that is $\operatorname{Re}\mu_j=0$. Moreover, under canonical transformations, F_{p_m} is transformed into a similar matrix and the eigenvalues of F_{p_m} remain invariant. The existence of non-vanishing real eigenvalues of $F_{p_m}(\hat{x},\hat{\xi})$ is a *necessary condition* for strong hyperbolicity. More precisely, let $p_{m-1}(x,\xi)=\sum_{|\alpha|=m-1}c_\alpha(x)\xi^\alpha$ and let $p'_{m-1}(x,\xi)=p_{m-1}(x,\xi)+\frac{\mathbf{i}}{2}\sum_{j=0}^n\frac{\partial^2 p_m}{\partial x_j\partial\xi_j}(x,\xi)$ be the subprincipal symbol of P, which is invariantly defined for $(x,\xi)\in\Sigma_1(p_m)$. It was proved in [4] that if $F_{p_m}(\hat{x},\hat{\xi})$ has no non-vanishing real eigenvalues, that $\operatorname{Im} p'_{m-1}(\hat{x},\hat{\xi})=0$ is a necessary condition for (WCP). A hyperbolic operator with principal symbol $p_m(x,\xi)$ will be called *effectively hyperbolic* if at every point $(\hat{x},\hat{\xi})\in\Sigma_1(p_m)$, the fundamental matrix $F_{p_m}(\hat{x},\hat{\xi})$ has two non-vanishing real eigenvalues. V. Ivrii introduced the following.

Conjecture. *A hyperbolic operator is strongly hyperbolic if and only if it is effectively hyperbolic.*

For operators with at most double characteristics the sufficient part of the above conjecture has been established in a number of papers: see [5,7] for two different proofs as well as [3,6,8] for some special classes of operators. According to Theorem 3 and Corollary 3 in [4], a strongly hyperbolic operator may have triple characteristics only for $t=0$ or $t=T$ and in this Note we deal with the case when this happens for $t=0$. More precisely, we examine operators having the form:

$$P=D_t^3+\sum_{j=1}^3 q_j(t,x,D_x)D_t^{3-j}+\sum_{j=1}^2 r_j(t,x,D_x)D_t^{2-j}+\sum_{j=1}^2 r_j(t,x)D_t^j+m_1(t,x,D_x)+c_0(t,x).$$

Here $q_j(t,x,D_x)$, $j=1,2,3$, are differential operators with C^∞ coefficients and real-valued symbols $q_j(t,x,\xi)$ that are homogeneous polynomials of order j in ξ, $r_j(t,x,D_x)$, $j=1,2$, are differential operators with C^∞ coefficients and symbols $r_j(t,x,\xi)$ homogeneous of order j with respect to ξ, $m_1(t,x,D_x)$ is a first order differential operator with C^∞ coefficients and $r_j(t,x)$, $c_0(t,x)$ are C^∞ functions. Consider the symbols $\Delta_1=27q_3-9q_1q_2+2q_1^3$, $\Delta_0=q_1^2-3q_2$, $\Delta=-\frac{1}{27}(\Delta_1^2-4\Delta_0^3)$. The symbol Δ is the *discriminant* of the equation $p_3=0$ with respect to τ and we have three real roots for $t\geqslant 0$ if and only if $\Delta\geqslant 0$. The symbol Δ_0 is the *discriminant* of the equation $\partial_\tau p_3=0$ and if we have a triple root at $\rho=(0,x_0,\xi)$, we get $\Delta_0(\rho)=\Delta_1(\rho)=0$. Moreover, the hyperbolicity of p_3 for $t\geqslant 0$ implies $d_{t,x}\Delta_1(\rho)=0$.

We make the following assumptions:

(H_0) The roots of the equation $p_3(t,x,\tau,\xi)=0$ with respect to τ are real for all $(t,x)\in\bar{G}$, $\xi\in\mathbb{R}^n\setminus\{0\}$.
(H_1) If the equation $p_3(0,x,\tau,\xi)=0$ with respect to τ has a triple root $\tau=\lambda(0,x_0,\xi_0)$ for $t=0$, $x_0\in U$, $\xi_0\in\mathbb{R}^n\setminus\{0\}$, then we have a triple root $\tau=\lambda(0,x_0,\xi)$ for $(0,x_0,\xi)$, $\forall\xi\in\mathbb{R}^n\setminus\{0\}$ and the fundamental matrix $F_{p_3}(0,x_0,\lambda(0,x_0,\xi),\xi)$ of p_3 has non-zero real eigenvalues $\pm\mu(x_0,\xi)$ for $\xi\in\mathbb{R}^n\setminus\{0\}$.
(H_2) If (H_1) holds for $(0,x_0,\xi)$, then there exists an open neighborhood U_{x_0} of x_0 such that $\Delta_1(0,x,\xi)=(\partial_t\Delta_1)(0,x,\xi)=0$ for $x\in U_{x_0}$ and $\xi\in\mathbb{R}^n\setminus\{0\}$.

Our main result is the following.

Theorem 3. *Assume the hypothesis (H_0)–(H_2) satisfied. Then there exists a open neighborhood $V_{x_0}\subset U_{x_0}$ such that for $T>0$ sufficiently small, the operator P is strongly hyperbolic in $\{(t,x)\colon\ 0\leqslant t\leqslant T, x\in V_{x_0}\}$.*

For operators with triple characteristics, the conjecture of Ivrii has been proved in [3] in the case when the principal symbol p_3 of P admits a factorization $p_3=((\tau-\beta(t,x,\xi))^2-D(t,x,\xi))(\tau-\gamma(t,x,\xi))$ with smooth real valued symbols β,γ homogeneous of order 1 in ξ and real-valued symbol $D\geqslant 0, t\geqslant 0$ homogeneous of order 2 in ξ. On the other hand, it was proved in [1] that for symbols having the form (2.2) given below with $b_3(0,x_0,\xi_0)\neq 0$, a factorization in a neighborhood of $(0,x_0)$ is possible if and only if $\alpha(x,\xi_0)\equiv 0$ for x in a neighborhood of x_0.

2. Idea of the proof of Theorem 3

To prove Theorem 3, we obtain an a priori estimate with a loss of regularity of order $2N/3-2$ for the operator P. We choose $N=\frac{13}{2}\Pi+N_0$, with N_0 an integer and:

$$\Pi=\frac{2}{3}+\sup_{x\in\bar{U}_{x_0},\ |\xi|=1}\left|p_2'\big(0,x,\lambda(x,\xi),\xi\big)\big(\mu(x,\xi)\big)^{-1}\right|,\tag{2.1}$$

where $p_2'(t,x,\tau,\xi)$ is the subprincipal symbol of P, $\lambda(0,x,\xi)$ denotes the triple root of $p_3=0$ and $\mu(x,\xi)$ is the non-vanishing eigenvalue of F_{p_3}. Moreover, the integer N_0 depends only on $p_3(0,x,\lambda(0,x,\xi),\xi)$, but we are not going to clarify the optimal value of N_0.

Step 1. By a change of variables (t,x), we reduce the analysis to the case when the principal symbol has the form:

$$p_3(t,x,\tau,\xi)=\tau^3-\big(ta_2(t,x,\xi)+\alpha(x,\xi)\big)\tau+t^2b_3(t,x,\xi),\tag{2.2}$$

where $a_2(t,x,\xi),\alpha(x,\xi)$, are real valued symbols homogeneous with respect to ξ of order 2 and $a_2(t,x,\xi)\geqslant\delta|\xi|^2,\delta>0$, $\alpha(x,\xi)\geqslant 0$, while $b_3(t,x,\xi)$ is homogeneous of order 3 in ξ. Next, we introduce the scaling $t=\epsilon^{2/3}s, x=\epsilon y, \epsilon>0$, and we transform P into an operator $\mathcal{P}$ with respect to (s,y). Since we are interested in showing that the Cauchy problem is well posed for sufficiently small t and since P is strictly hyperbolic for t positive and small enough, we can investigate the operator $\mathcal{P}$. We use again the notation (t,x) for the new variables. The symbols a_2, α, b_3 are transformed to symbols $a_2^\epsilon=a_2(\epsilon^{2/3}t,\epsilon x,\xi)$, $\alpha^\epsilon=\alpha(\epsilon x,\xi)$, $b_3^\epsilon=b_3(\epsilon^{2/3}t,\epsilon x,\xi)$ and this is important for the pseudodifferential calculus which we develop. Eventually we choose $\epsilon=\mathcal{O}(N^{-1})$, where N is given above, so that $0<\epsilon_0\leqslant\epsilon N\ll 1$.

Step 2. We choose a time function $f=\frac{t}{3}+\langle\xi\rangle^{-2/3}$, $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$ which plays an important role in the calculus of pseudodifferential operators with order function $m_N^t=f^{-N}(t,\xi)$ and metric $g^\epsilon_{(x,\xi)}=\epsilon^2|dx|^2+\langle\xi\rangle^{-2}|d\xi|^2$. The issue of this choice is that the commutator $[a_2^\epsilon(t,x,D_x),f^{-N}(t,D_x)]$ has symbol in the space $S(f^{-N}\epsilon N\langle\xi\rangle,g^\epsilon)$, so $[a_2^\epsilon(t,x,D_x),f^{-N}(t,D_x)]f^N(t,D_x)$ becomes a first-order operator whose norm is not depending on N and, in particular, on Π.

Step 3. In order to derive energy estimates, we multiply $\mathcal{P}u$ by the *multiplier* operator:

$$Mu=\psi(t)\left(D_t^2u-\frac{1}{3}\big(ta_2^\epsilon(t,x,D_x)u+\epsilon^{-2/3}\alpha^\epsilon(x,D_x)u\big)\right),$$

where $\psi(t)=\frac{e^{-2\lambda t}}{t}$, $\lambda>0$, and then study the expression $-2\operatorname{Im}\langle f^{-2N}(t,D_x)\mathcal{P}u,Mu\rangle$, $\langle,\rangle$ denoting the scalar product in $L^2(\mathbb{R}^n)$ and $u\in C_0^\infty([0,T]\times U)$. The above expression modulo lower-order terms is a sum of fifteen terms and we make a quite detailed analysis of all these terms. The purpose is to find, by integration by parts, "positive" terms with a big coefficient of order $\mathcal{O}(N)$ which will absorb in the energy estimate the contributions with "indefinite" (possibly negative) sign.

Step 4. The positive terms in the energy estimates come from the expression $\partial_t\mathcal{E}_N(u)+\frac{2N}{3}\mathcal{E}_{N+1/2}(u)$, where:

$$\mathcal{E}_N(u) = \psi\Big[\big\|f^{-N}u''\big\|_0^2 + \frac{2}{3}\mathrm{Re}\big\langle f^{-2N}\big(ta_2^\epsilon + \epsilon^{-2/3}\alpha^\epsilon\big)u', u'\big\rangle + \frac{1}{3}\big\|f^{-N}\big(ta_2^\epsilon u + \epsilon^{-2/3}\alpha^\epsilon\big)u\big\|_0^2$$
$$+ \frac{2}{3}\mathrm{Re}\big\langle f^{-2N}u'', \big(ta_2^\epsilon + \epsilon^{-2/3}\alpha^\epsilon\big)u\big\rangle\Big] + \epsilon^{1/3}te^{-2\lambda t}2\,\mathrm{Im}\big\langle f^{-2N}b_3^\epsilon u, u'\big\rangle. \tag{2.3}$$

The quantity in the second line above, involving powers of ϵ and t, is a lower-order perturbation.

Step 5. To control the terms involving $\|f^{-k}u(t,.)\|_2$ norms and $k = N, N+1/2$, we would like to exploit the terms with $H^2(\mathbb{R}^n)$ norms having large coefficients. However, the latter terms have factors t so that tN becomes close to 0. To overcome this difficulty, we use the key inequality $\langle\xi\rangle^4 f^{-k} \leqslant t\langle\xi\rangle^4 f^{-k-1} + \langle\xi\rangle^2 f^{-k-3}$. Thus, for example, we have an estimate $\|f^{-N}u\|_1^2 \leqslant t\|f^{N-1/2}u\|_1^2 + \|f^{-N-3/2}u\|_0^2$. For $\|f^{-N+1/2}u\|_2^2$, we must apply the key inequality twice to gain t^2 factors and this leads to norms of the form $\|f^{-N-5/2}u\|_0$. Following this method, we also obtain negative terms with coefficients t and t^2, as well as a number of other negative terms involving norms with *weights* $f^{-N-3/2}$ and $f^{-N-5/2}$; these have to be absorbed too. To be able to handle such terms, we use some differential inequalities involving ψ and $-\psi'$ and we take the parameter λ sufficiently large. Finally, we obtain:

Theorem 4. *Assume that $(D_t^2u)(t,x) = (D_tu)(t,x) = u(t,x) = 0$ and let $0 \leqslant t \leqslant s \leqslant T$ with a small $T > 0$. Then for every $p \in \mathbb{R}$ there exist Λ_p and a constant C_p so that for $\lambda \geqslant \Lambda_p$, $N = \frac{13}{2}\Pi + N_0$ and $u \in C_0^\infty(\mathbb{R}^{n+1})$, we have the estimate*:

$$\lambda\int_t^T e^{-2\lambda s - 2N\log(1+s)}\left(\sum_{k=0}^{2} s^{1-k}\big\|\partial_t^k u(s,.)\big\|_{(2-k+p)}^2 + \sum_{k=0}^{1}\big\|\partial_t^k u(s,.)\big\|_{(1-k+p)}^2\right)\mathrm{d}s$$
$$\leqslant C_p\int_t^T e^{-2\lambda s - 2N\log(1+\frac{s}{3})}\big\|Pu(s,.)\big\|_{(2N/3+p)}^2\,\mathrm{d}s, \tag{2.4}$$

where $\|.\|_{(m)}$ is the H^m norm in $\mathbb{R}^n$.

The details of the proofs are included in [1].

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