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Sym–Bobenko formula for minimal surfaces in Heisenberg space ☆

Une formule de Sym–Bobenko pour les surfaces minimales dans l'espace de Heisenberg

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ABSTRACT

We give an immersion formula, the *Sym–Bobenko formula*, for minimal surfaces in the 3-dimensional Heisenberg space. Such a formula can be used to give a generalized Weierstrass type representation and construct explicit examples of minimal surfaces.

RÉSUMÉ

On donne une formule d'immersions, dite de *Sym–Bobenko*, pour les surfaces minimales de l'espace de Heisenberg de dimension 3. Une telle formule peut être utilisée pour écrire une représentation de Weierstrass généralisée et construire des exemples explicites de surfaces minimales.

1. Introduction

A Sym–Bobenko formula is the expression of an immersion in terms of a one-parameter family of moving frames, called the *extended frame*. This idea was first used by A. Sym [9] in the case of surfaces with negative constant (Gauss) curvature in Euclidean space. A.I. Bobenko applied the method to numerous cases [1–3], including constant mean curvature (*CMC* for short) surfaces in space forms—Euclidean 3-space, 3-sphere and hyperbolic 3-space—and T. Taniguchi applied it to CMC spacelike surfaces in Minkowski 3-space [10]. In the meantime, the works of J. Dorfmeister, F. Pedit and H. Wu [8] and of D. Brander, W. Rossman and N. Schmitt [4] show that the Sym–Bobenko formulas can be seen as generalized Weierstrass-type representations for CMC surfaces, extended frames coming from holomorphic data.

In Heisenberg's 3-space, the classical method does not apply, since the isometry group is of dimension only 4—contrary to the ones of space forms that are 6-dimensional—and does not act transitively on orthonormal frames; there are "not enough" isometries to define a moving frame. We show that, nevertheless, for minimal immersions, a Sym–Bobenko formula can be established using an *ad-hoc* matrix-valued map.

☆ This work is part of the author's Ph.D. thesis [5].

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In [7], J.F. Dorfmeister, J. Inoguchi and S. Kobayashi link this formula with pairs of meromorphic and anti-meromorphic 1-forms, which they call *pairs of normalized potentials*, in a way to get a generalized Weierstrass-type representation for minimal surfaces.

2. Surfaces in Heisenberg space

We see the 3-dimensional Heisenberg space Nil_3 as $\mathbb{R}^3$, with generic coordinates (x_1, x_2, x_3), endowed with the following Riemannian metric:

$$\langle\cdot,\cdot\rangle = dx_1^2 + dx_2^2 + \left(\frac{1}{2}(x_2\,dx_1 - x_1\,dx_2) + dx_3\right)^2.$$

We call *canonical frame* the orthonormal frame (E_1, E_2, E_3) defined by:

$$E_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial x_3}, \qquad E_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial x_3} \quad \text{and} \quad E_3 = \frac{\partial}{\partial x_3},$$

and the Levi-Civita connection ∇ writes:

$$\begin{aligned}
&\nabla_{E_1} E_1 = 0, && \nabla_{E_2} E_1 = -\frac{1}{2}E_3, && \nabla_{E_3} E_1 = -\frac{1}{2}E_2,\\
&\nabla_{E_1} E_2 = \frac{1}{2}E_3, && \nabla_{E_2} E_2 = 0, && \nabla_{E_3} E_2 = \frac{1}{2}E_1,\\
&\nabla_{E_1} E_3 = -\frac{1}{2}E_2, && \nabla_{E_2} E_3 = \frac{1}{2}E_1, && \nabla_{E_3} E_3 = 0.
\end{aligned}$$

Note that E_3 is a Killing field and that the projection $\pi : (x_1, x_2, x_3) \in \mathrm{Nil}_3 \mapsto (x_1, x_2) \in \mathbb{R}^2$ on the first two coordinates is a Riemannian submersion. From now on, we identify $\mathbb{R}^2$ with $\mathbb{C}$.

We may also write Nil_3 as a subset of $\mathcal{M}_2(\mathbb{C})$. Consider the matrices:

$$\sigma_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The identification is the following:

$$(x_1, x_2, x_3) \in \mathrm{Nil}_3 \quad \longleftrightarrow \quad x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 = \begin{pmatrix} x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_3 \end{pmatrix} \in \mathcal{M}_2(\mathbb{C}). \tag{1}$$

Note that this identification is purely formal and does not involve any manifold-related structure.

Let Σ be a simply connected Riemann surface and z be a conformal parameter on Σ. A conformal immersion is denoted by $f : \Sigma \to \mathrm{Nil}_3$ with unit normal N and conformal factor $\rho : \Sigma \to (0, +\infty)$ meaning:

$$\langle f_z, f_z\rangle = \langle f_{\bar z}, f_{\bar z}\rangle = 0, \qquad \langle f_z, f_{\bar z}\rangle = \frac{\rho}{2}, \qquad \langle f_z, N\rangle = \langle f_{\bar z}, N\rangle = 0 \quad \text{and} \quad \langle N, N\rangle = 1.$$

Let $\varphi = \langle N, E_3\rangle : \Sigma \to (-1, 1)$ denote the angle function of N, $A = \langle f_z, E_3\rangle : \Sigma \to \mathbb{C}$ the vertical part of f_z and $p\,dz^2 = \langle\nabla_{f_z} f_z, N\rangle\,dz^2$ the Hopf differential of f.

The Abresch–Rosenberg quadratic differential expresses $Q\,dz^2 = (ip + A^2)\,dz^2$, and a necessary and sufficient condition for f to be minimal is $\nabla_{f_z} f_{\bar z} = 0$.

We also decompose f into $f = (F, h)$ with $F = \pi \circ f : \Sigma \to \mathbb{C}$ the horizontal projection of f and $h : \Sigma \to \mathbb{R}$ its height function. We can express A in terms of F and h:

$$A = h_z - \frac{i}{4}(F\overline{F}_z - \overline{F}F_z). \tag{2}$$

In the matrix model (1) of Nil_3, the map F is given by the non-diagonal coefficients—precisely the $(1, 2)$-coefficient—and h by the diagonal ones.

The intuitive idea behind the Sym–Bobenko formulas in space forms is that, up to ambient isometries, the unit normal—or the Gauss map—would locally determine the immersion up to ambient isometries. In Nil_3, such a map is defined as follows; see [6] for details. Since Nil_3 is a Lie group, the map $f^{-1}N$ takes values in the unit sphere $\mathbb{S}^2$ of the Lie algebra. Moreover, for a local study, we can suppose $\varphi > 0$ so that the values of $f^{-1}N$ are actually in the northern hemisphere of $\mathbb{S}^2$. If s denotes the stereographic projection centered at the South Pole, we call *Gauss map* of an immersion f the map $g = s \circ (f^{-1}N)$ with values in the unit disk. Actually, endowing the unit disk with the Poincaré metric, we see the Gauss map g as a map with values into the hyperbolic disk $\mathbb{H}^2$.

We use the following criterion to show that a conformal immersion in the Heisenberg space is minimal:

Proposition 2.1. *(See Daniel [6].) A conformal immersion $f = (F, h) : \Sigma \to \mathrm{Nil}_3$ is minimal if and only if:*

$$F_{z\bar{z}} = \frac{\mathrm{i}}{2}(\bar{A}F_z + AF_{\bar{z}}) \quad \text{and} \quad A_{\bar{z}} + \bar{A}_z = 0.$$

Furthermore, when f is minimal its Gauss map $g : \Sigma \to \mathbb{H}^2$ is harmonic.

3. The Sym–Bobenko formula

Consider the family $(\Psi_t)_{t\in\mathbb{R}}$ of matrix fields over Σ which are solutions of the system:

$$\begin{cases} \Psi_t^{-1}\mathrm{d}\Psi_t = \dfrac{1}{4}\begin{pmatrix} (\log\rho_0)_z & \mathrm{i}\sqrt{\rho_0} \\ -\frac{4\mathrm{i}Q_0}{\sqrt{\rho_0}}\mathrm{e}^{2\mathrm{i}t} & -(\log\rho_0)_z \end{pmatrix}\mathrm{d}z + \dfrac{1}{4}\begin{pmatrix} -(\log\rho_0)_{\bar{z}} & \frac{4\mathrm{i}\overline{Q_0}}{\sqrt{\rho_0}}\mathrm{e}^{-2\mathrm{i}t} \\ -\mathrm{i}\sqrt{\rho_0} & (\log\rho_0)_{\bar{z}} \end{pmatrix}\mathrm{d}\bar{z} \\ \Psi_t(z=0) = \sigma_3 \end{cases}$$

where $\rho_0 : \Sigma \to (0, +\infty)$ and $Q_0 : \Sigma \to \mathbb{C}$ are smooth. Such a family (Ψ_t) exists if and only if:

$$(\log\rho_0)_{z\bar{z}} = \frac{\rho_0}{8} - \frac{2|Q_0|^2}{\rho_0} \quad \text{and} \quad (Q_0)_{\bar{z}} = 0.$$

Theorem 3.1 *(Sym–Bobenko formula). Using the matrix model (1), define the map $f_t : \Sigma \to \mathrm{Nil}_3$ for any $t \in \mathbb{R}$ as:*

$$f_t = -\frac{1}{2}\left(\sigma_0\frac{\partial\widehat{f}_t}{\partial t}\right)^{\mathrm{d}} + (\widehat{f}_t)^{\mathrm{nd}} \quad \text{with } \widehat{f}_t = -2\frac{\partial\Psi_t}{\partial t}\Psi_t^{-1} + 2\Psi_t\sigma_0\Psi_t^{-1}, \tag{3}$$

where the superscripts $\cdot^{\mathrm{d}}$ and $\cdot^{\mathrm{nd}}$ denote respectively the diagonal and non-diagonal terms. Then f_t is a conformal minimal immersion in the Heisenberg space and the family (f_t) is the so-called associated family.

Proof. Fix $t \in \mathbb{R}$. From Eq. (3), we get:

$$(f_t)^{\mathrm{nd}} = (\widehat{f}_t)^{\mathrm{nd}}, \qquad \big((f_t)_z\big)^{\mathrm{d}} = \frac{1}{2}\big(\sigma_0\big[(\widehat{f}_t)_z, \widehat{f}_t\big]\big)^{\mathrm{d}} + 2\big(\sigma_0(\widehat{f}_t)_z\big)^{\mathrm{d}} \quad \text{and} \quad (\widehat{f}_t)_{z\bar{z}} = \frac{\mathrm{i}}{4}\big[(\widehat{f}_t)_z, (\widehat{f}_t)_{\bar{z}}\big], \tag{4}$$

where $[\cdot,\cdot]$ denotes the commutator. From the first equation in (4), we have that matrices f_t and $\widehat{f}_t$ write:

$$f_t = \begin{pmatrix} h & F \\ \bar{F} & h \end{pmatrix} \quad \text{and} \quad \widehat{f}_t = \begin{pmatrix} \mathrm{i}\widehat{h} & F \\ \bar{F} & -\mathrm{i}\widehat{h} \end{pmatrix},$$

with $F : \Sigma \to \mathbb{C}$ and $h, \widehat{h} : \Sigma \to \mathbb{R}$ smooth. We show that F and h verify the conditions of Proposition 2.1. Using Eq. (2) and the second identity in (4), we deduce $A = \mathrm{i}\widehat{h}_z$ and since $\widehat{h}$ is real-valued, we obtain $A_{\bar{z}} + \bar{A}_z = 0$. Finally, the $(1,2)$-coefficient of the third equation in (4) verifies:

$$F_{z\bar{z}} = \frac{1}{2}(\widehat{h}_{\bar{z}}F_z - \widehat{h}_zF_{\bar{z}}) = \frac{\mathrm{i}}{2}(\bar{A}F_z + AF_{\bar{z}}),$$

which concludes the proof. □

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