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Green's formulas with little regularity on a surface – Application to Donati-like compatibility conditions on a surface

Formules de Green avec peu de régularité sur une surface – Application à des conditions de compatibilité du type de Donati sur une surface

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A R T I C L E   I N F O

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A B S T R A C T

In this Note, we establish two Green's formulas with little regularity on a surface. These formulas are then used for identifying and justifying Donati-like compatibility conditions on a surface, guaranteeing that the components of two symmetric matrix fields $(c_{\alpha\beta})$ and $(r_{\alpha\beta})$ with $c_{\alpha\beta}$ and $r_{\alpha\beta}$ in the space $L^2(\omega)$, where ω is a domain in $\mathbb{R}^2$, are the covariant components of the linearized change of metric and linearized change of curvature tensors associated with a displacement vector field of a surface $\boldsymbol{\theta}(\overline{\omega})$, where $\boldsymbol{\theta} : \overline{\omega} \to \mathbb{R}^3$ is a smooth immersion.

R É S U M É

Dans cette Note, on établit deux formules de Green avec peu de régularité sur une surface. Ces formules sont ensuite utilisées pour identifier et justifier des conditions de compatibilité du type de Donati sur une surface, garantissant que les composantes de deux champs de matrices symétriques $(c_{\alpha\beta})$ et $(r_{\alpha\beta})$ avec $c_{\alpha\beta}$ et $r_{\alpha\beta}$ dans l'espace $L^2(\omega)$, où ω est un domaine ω de $\mathbb{R}^2$, sont les composantes covariantes des champs de tenseurs de changement de métrique et de changement de courbure linéarisés associés à un champ de déplacements d'une surface $\boldsymbol{\theta}(\overline{\omega})$, où $\boldsymbol{\theta} : \overline{\omega} \to \mathbb{R}^3$ est une immersion régulière.

1. Notations and geometrical preliminaries

Latin indices vary in the set $\{1, 2, 3\}$ save when they are used for indexing sequences, Greek indices vary in the set $\{1, 2\}$, and the summation convention with respect to repeated indices is systematically used in conjunction with these rules. The notation V' designates the dual space of V and ${}_{V'}\langle \cdot, \cdot \rangle_V$ denotes the duality between V' and V.

Let Ω be an open subset of $\mathbb{R}^N$. Spaces of functions, vector fields, and *symmetric* matrix fields, defined over Ω are respectively denoted by italic capitals, boldface italic capitals, and special Roman capitals. The Euclidean inner product and the vector product of $\boldsymbol{a} \in \mathbb{R}^N$ and $\boldsymbol{b} \in \mathbb{R}^N$ are denoted $\boldsymbol{a} \cdot \boldsymbol{b}$ and $\boldsymbol{a} \wedge \boldsymbol{b}$; the Euclidean norm of $\boldsymbol{a} \in \mathbb{R}^N$ is denoted $|\boldsymbol{a}|$.

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For any vector field $\boldsymbol{v} = (v_i) \in \boldsymbol{D}'(\Omega)$, the symmetric matrix field $\nabla_s \boldsymbol{v} \in \mathbb{D}'(\Omega)$ is defined by:

$$\nabla_s \boldsymbol{v} := \frac{1}{2}\left(\nabla \boldsymbol{v}^T + \nabla \boldsymbol{v}\right), \quad \text{or equivalently, by} \quad (\nabla_s \boldsymbol{v})_{ij} = \frac{1}{2}(\partial_i v_j + \partial_j v_i),$$

where ∇_s designates the *symmetrized gradient operator*.

A *domain in* $\mathbb{R}^2$ is a bounded, connected, open subset ω of $\mathbb{R}^2$ whose boundary γ is Lipschitz-continuous, the set ω being locally on the same side of γ. A measure, denoted $d\gamma$, can then be defined along γ and a unit outer normal vector (ν_α) exists $d\gamma$-almost everywhere along γ. The outer normal derivative operator $\partial_\nu := \nu_\alpha \partial_\alpha$ is thus defined $d\gamma$-almost everywhere along γ.

For the various notions of, and results from, differential geometry of surfaces used below, see, e.g., Chapters 2 and 4 in [5]. Let ω be a domain in $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$ be an *immersion*. Then the two vectors $\boldsymbol{a}_\alpha(y) := \partial_\alpha \boldsymbol{\theta}(y)$ form the covariant basis of the tangent plane to the *surface* $S := \boldsymbol{\theta}(\overline{\omega})$ at the point $\boldsymbol{\theta}(y)$ and the tangent vectors $\boldsymbol{a}^\beta(y)$ defined by $\boldsymbol{a}_\alpha(y) \cdot \boldsymbol{a}^\beta(y) = \delta_\alpha^\beta$ form the contravariant basis of the tangent plane to S at the same point.

The covariant components $a_{\alpha\beta} \in \mathcal{C}^2(\overline{\omega})$ and $b_{\alpha\beta} \in \mathcal{C}^1(\overline{\omega})$ of the *first*, and *second*, *fundamental forms* of the surface S are then respectively defined by $a_{\alpha\beta} := \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta$ and $b_{\alpha\beta} := \boldsymbol{a}^3 \cdot \partial_\alpha \boldsymbol{a}_\beta$, where, at each point $y \in \overline{\omega}$, $\boldsymbol{a}_3(y) = \boldsymbol{a}^3(y) := \frac{\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)}{|\boldsymbol{a}_1(y) \wedge \boldsymbol{a}_2(y)|}$ denotes a unit vector normal to the surface S at the point $\boldsymbol{\theta}(y)$. The contravariant components $a^{\alpha\beta} \in \mathcal{C}^2(\overline{\omega})$ of the first fundamental form and the mixed components $b_\alpha^\sigma \in \mathcal{C}^1(\overline{\omega})$ of the second fundamental form are then defined by $a^{\alpha\beta} := \boldsymbol{a}^\alpha \cdot \boldsymbol{a}^\beta$ and $b_\alpha^\sigma := a^{\sigma\beta} b_{\alpha\beta}$. The *Christoffel symbols* $\Gamma_{\alpha\beta}^\sigma \in \mathcal{C}^1(\overline{\omega})$ associated with the immersion $\boldsymbol{\theta}$ are defined by $\Gamma_{\alpha\beta}^\sigma := \boldsymbol{a}^\sigma \cdot \partial_\alpha \boldsymbol{a}_\beta$.

Two tensor fields play a fundamental role in the two-dimensional theory of linearly elastic shells: Given a vector field $\boldsymbol{\eta} = (\eta_i) : \overline{\omega} \to \mathbb{R}^3$, the *linearized change of metric tensor field* and the *linearized change of curvature tensor field* associated with the *displacement vector field* $\eta_i \boldsymbol{a}^i$ *of the surface* S are respectively defined by:

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) := \frac{1}{2}(\eta_{\alpha|\beta} + \eta_{\beta|\alpha}) - b_{\alpha\beta}\eta_3,$$
$$\rho_{\alpha\beta}(\boldsymbol{\eta}) := \eta_{3|\alpha\beta} - b_\alpha^\sigma b_{\sigma\beta}\eta_3 + \left(b_\alpha^\sigma\right)\eta_{\sigma|\beta} + \left(b_\beta^\tau\right)\eta_{\tau|\alpha} + \left(b_{\beta|\alpha}^\tau\right)\eta_\tau, \tag{1}$$

where:

$$\eta_{\beta|\alpha} := \partial_\alpha \eta_\beta - \Gamma_{\alpha\beta}^\sigma \eta_\sigma, \qquad \eta_{3|\alpha\beta} := \partial_{\alpha\beta}\eta_3 - \Gamma_{\alpha\beta}^\sigma \partial_\sigma \eta_3, \qquad b_{\beta|\alpha}^\tau := \partial_\alpha b_\beta^\tau + \Gamma_{\alpha\sigma}^\tau b_\beta^\sigma - \Gamma_{\alpha\beta}^\sigma b_\sigma^\tau. \tag{2}$$

Note that the above definitions of the components $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})$ are meant to hold either *in the usual sense* or *in the sense of distributions*.

Complete proofs, and applications to intrinsic linear shell theory, will be found in [6].

2. A fundamental Green's formula

The following Green's formula is the basis of our approach for deriving compatibility conditions of Donati type. Note that it is implicitly understood in the statement and proof of Theorem 2.1 that functions such as $\gamma_{\alpha\beta}(\boldsymbol{\eta})$, $\sqrt{a}$, $b_{\alpha\beta}$, etc., correspond to the immersion $\boldsymbol{\theta}$ appearing in it; the same observation applies throughout this Note.

Theorem 2.1. *Let there be given a domain* $\omega \subset \mathbb{R}^2$ *with a boundary* γ *of class* $\mathcal{C}^{1,1}$ *and an immersion* $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$. *Then the following* Green's formula *holds for all tensor fields* $\boldsymbol{n} = (n^{\alpha\beta}) \in \mathbb{H}^1(\omega)$ *and* $\boldsymbol{m} = (m^{\alpha\beta}) \in \mathbb{H}^2(\omega)$ *and for all vector fields* $\boldsymbol{\eta} = ((\eta_\alpha), \eta_3) \in \boldsymbol{H}^1(\omega) \times H^2(\omega)$:

$$\int_\omega \left(n^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{n}) + m^{\alpha\beta}\rho_{\alpha\beta}(\boldsymbol{\eta})\right)\sqrt{a}\,dy + \int_\omega d^i(\boldsymbol{n}, \boldsymbol{m})\eta_i\sqrt{a}\,dy = \int_\gamma \left\{b^i(\boldsymbol{n}, \boldsymbol{m})\eta_i + b^\nu(\boldsymbol{n}, \boldsymbol{m})\partial_\nu \eta_3\right\}\sqrt{a}\,d\gamma, \tag{3}$$

where the covariant components $\gamma_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega)$ *and* $\rho_{\alpha\beta}(\boldsymbol{\eta}) \in L^2(\omega)$ *are defined in* (1)–(2), *and the functions*

$$d^i(\boldsymbol{n}, \boldsymbol{m}) \in L^2(\omega), \qquad b^i(\boldsymbol{n}, \boldsymbol{m}) \in L^2(\gamma), \qquad b^\nu(\boldsymbol{n}, \boldsymbol{m}) \in L^2(\gamma),$$

are defined by

$$d^\alpha(\boldsymbol{n}, \boldsymbol{m}) := \left(n^{\alpha\beta} + b_\sigma^\alpha m^{\sigma\beta}\right)\big|_\beta + b_\sigma^\alpha\left(m^{\beta\sigma}\big|_\beta\right), \qquad d^3(\boldsymbol{n}, \boldsymbol{m}) := -m^{\alpha\beta}\big|_{\alpha\beta} + b_\alpha^\sigma b_{\sigma\beta} m^{\alpha\beta} + b_{\alpha\beta} n^{\alpha\beta}, \tag{4}$$

$$b^\alpha(\boldsymbol{n}, \boldsymbol{m}) := \left(n^{\alpha\beta} + 2b_\sigma^\alpha m^{\alpha\beta}\right)\nu_\beta, \qquad b^3(\boldsymbol{n}, \boldsymbol{m}) := -m^{\alpha\beta}\big|_\beta \nu_\alpha - \partial_\tau\left(m^{\alpha\beta}\nu_\alpha\tau_\beta\right), \qquad b^\nu(\boldsymbol{n}, \boldsymbol{m}) := m^{\alpha\beta}\nu_\alpha\nu_\beta, \tag{5}$$

where, for any smooth enough symmetric tensor field $(t^{\alpha\beta})$,

$$t^{\alpha\beta}\big|_\beta := \partial_\beta t^{\alpha\beta} + \Gamma_{\beta\sigma}^\alpha t^{\beta\sigma} + \Gamma_{\sigma\tau}^\tau t^{\alpha\sigma} \quad \text{and} \quad t^{\alpha\beta}\big|_{\alpha\beta} := \partial_\alpha\left(t^{\alpha\beta}\big|_\beta\right) + \Gamma_{\alpha\sigma}^\sigma\left(t^{\alpha\beta}\big|_\beta\right).$$

Proof. The proof simply relies on repeated applications of Green's formulas in Sobolev spaces and on the relations $\partial_\beta \sqrt{a} = \sqrt{a}\Gamma_{\beta\tau}^\tau$; see the proofs of Theorems 4.5-1 and 7.1-3 in [4]. Note that the assumption that γ is of class $\mathcal{C}^{1,1}$ ensures that $b^3(\boldsymbol{n}, \boldsymbol{m}) \in L^2(\gamma)$. $\square$

3. The operator $(\boldsymbol{\gamma}, \boldsymbol{\rho})$

The linear operator $(\boldsymbol{\gamma}, \boldsymbol{n})$ is defined by:

$$\big(\boldsymbol{\gamma}(\boldsymbol{n}), \boldsymbol{\rho}(\boldsymbol{\eta})\big) := \big(\big(\gamma_{\alpha\beta}(\boldsymbol{\eta})\big), \big(\rho_{\alpha\beta}(\boldsymbol{n})\big)\big)$$

for smooth enough vector fields $\boldsymbol{\eta} = (\eta_i) : \overline{\omega} \to \mathbb{R}^3$, where $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})$ are defined in (1)–(2). To begin with, we consider the case where the operator $(\boldsymbol{\gamma}, \boldsymbol{\eta})$ is considered as acting from the space $H_0^1(\omega) \times H_0^1(\omega) \times H_0^2(\omega)$ into the space $\mathbb{L}^2(\omega) \times \mathbb{L}^2(\omega)$.

Theorem 3.1. *Let there be given a domain $\omega \subset \mathbb{R}^2$ with a boundary γ of class $\mathcal{C}^{1,1}$ and an immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$. Let*

$$\boldsymbol{H}_0^1(\omega) := H_0^1(\omega) \times H_0^1(\omega) \quad \text{and} \quad \boldsymbol{H}^{-1}(\omega) := H^{-1}(\omega) \times H^{-1}(\omega).$$

(a) *The image of the space $\boldsymbol{H}_0^1(\omega) \times H_0^2(\omega)$ under the operator*:

$$(\boldsymbol{\gamma}, \boldsymbol{\rho}) : \boldsymbol{\eta} = \big((\eta_\alpha), \eta_3\big) \in \boldsymbol{H}_0^1(\omega) \times H_0^2(\omega) \to \big(\boldsymbol{\gamma}(\boldsymbol{n}), \boldsymbol{\rho}(\boldsymbol{n})\big) \in \mathbb{L}^2(\omega) \times \mathbb{L}^2(\omega)$$

is a closed subspace of the space $\mathbb{L}^2(\omega) \times \mathbb{L}^2(\omega)$.
(b) *The following* Green's formula with little regularity *holds for all $(\boldsymbol{n}, \boldsymbol{m}) = ((n^{\alpha\beta}), (m^{\alpha\beta})) \in \mathbb{L}^2(\omega) \times \mathbb{L}^2(\omega)$ and all $((\eta_\alpha), \eta_3) \in \boldsymbol{H}_0^1(\omega) \times H_0^2(\omega)$*:

$$\int_\omega \big(n^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{\eta}) + m^{\alpha\beta}\rho_{\alpha\beta}(\boldsymbol{\eta})\big)\sqrt{a}\, \mathrm{d}y + {}_{H^{-1}(\omega)}\big\langle \sqrt{a} d^\alpha(\boldsymbol{n}, \boldsymbol{m}), \eta_\alpha \big\rangle_{H_0^1(\omega)} + {}_{H^{-2}(\omega)}\big\langle \sqrt{a} d^3(\boldsymbol{n}, \boldsymbol{m}), \eta_3 \big\rangle_{H_0^2(\omega)} = 0, \tag{6}$$

where, for each $(\boldsymbol{n}, \boldsymbol{m}) \in \mathbb{L}^2(\omega) \times \mathbb{L}^2(\omega)$, the distributions $d^\alpha(\boldsymbol{n}, \boldsymbol{m}) \in H^{-1}(\omega)$ and $d^3(\boldsymbol{n}, \boldsymbol{m}) \in H^{-2}(\omega)$ are defined as in (4).
(c) *Let the space $\mathbb{L}^2(\omega) \times \mathbb{L}^2(\omega)$ be equipped with the inner product defined for any $((n^{\alpha\beta}), m^{\alpha\beta}) \in \mathbb{L}^2(\omega) \times \mathbb{L}^2(\omega)$ and any $((c_{\alpha\beta}), (r_{\alpha\beta})) \in \mathbb{L}^2(\omega) \times \mathbb{L}^2(\omega)$ by*:

$$\int_\omega \big\{n^{\alpha\beta}c_{\alpha\beta} + m^{\alpha\beta}r_{\alpha\beta}\big\}\sqrt{a}\, \mathrm{d}y.$$

Then the dual operator of $(\boldsymbol{\gamma}, \boldsymbol{\rho}) : \boldsymbol{H}_0^1(\omega) \times H_0^2(\omega) \to \mathbb{L}^2(\omega) \times \mathbb{L}^2(\omega)$ is:

$$(\boldsymbol{n}, \boldsymbol{m}) \in \mathbb{L}^2(\omega) \times \mathbb{L}^2(\omega) \to -\big(\big(\sqrt{a} d^\alpha(\boldsymbol{n}, \boldsymbol{m})\big), \sqrt{a} d^3(\boldsymbol{n}, \boldsymbol{m})\big) \in \boldsymbol{H}^{-1}(\omega) \times H^{-2}(\omega).$$

Sketch of proof. Part (a) is a consequence of the *Korn inequality "with boundary conditions" on a surface* due to Bernadou and Ciarlet [2] (see also Bernadou, Ciarlet and Miara [3]). Note that, like the classical "three-dimensional" Korn inequality, this Korn inequality essentially hinges on the fundamental *lemma of J.-L. Lions* (see Theorem 3.2 in Chapter 3 of Duvaut and Lions [9]).

The Green's formula (6) is then established like the fundamental Green's formula (3), but with the integral $\int_\omega d^i(\boldsymbol{n}, \boldsymbol{m})\eta_i\sqrt{a}\,\mathrm{d}y$ now replaced with appropriate duality brackets. Note that there are no duality brackets on the boundary γ to replace the integrals over γ, because $((\eta_\alpha), \eta_3) \in \boldsymbol{H}_0^1(\omega) \times H_0^2(\omega)$.

The property stated in (c) is an immediate consequence of the Green's formula of (b). □

We next consider the case where the operator $(\boldsymbol{\gamma}, \boldsymbol{\eta})$ is considered as acting from the space $\boldsymbol{L}^2(\omega) := L^2(\omega) \times L^2(\omega) \times L^2(\omega)$ into the space $\mathbb{H}^{-1}(\omega) \times \mathbb{H}^{-2}(\omega)$, in which case one can show that, like for smooth vector fields $\boldsymbol{\eta}$,

$$\begin{aligned}\mathbf{Ker}(\boldsymbol{\gamma}, \boldsymbol{\eta}) &:= \big\{\boldsymbol{\eta} \in \boldsymbol{L}^2(\omega);\ \boldsymbol{\gamma}(\boldsymbol{\eta}) = \mathbf{0} \text{ in } \mathbb{H}^{-1}(\omega) \text{ and } \boldsymbol{\rho}(\boldsymbol{\eta}) = \mathbf{0} \text{ in } \mathbb{H}^{-2}(\omega)\big\} \\ &= \big\{\boldsymbol{\eta} \in \boldsymbol{\mathcal{C}}^3(\overline{\omega});\ \text{there exist } \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3 \text{ such that } \eta_i(y)\boldsymbol{a}^i(y) = \boldsymbol{a} + \boldsymbol{b} \wedge \boldsymbol{\theta}(y),\ y \in \overline{\omega}\big\}.\end{aligned}$$

Theorem 3.2. *Let there be given a domain $\omega \subset \mathbb{R}^2$ with a boundary γ of class $\mathcal{C}^{1,1}$ and an immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{R}^3)$.*

(a) *The image of the space $\boldsymbol{L}^2(\omega) := L^2(\omega) \times L^2(\omega) \times L^2(\omega)$ under the operator*

$$(\boldsymbol{\gamma}, \boldsymbol{\rho}) : \boldsymbol{\eta} = (\eta_i) \in \boldsymbol{L}^2(\omega) \to \big(\gamma(\boldsymbol{\eta}), \rho(\boldsymbol{\eta})\big) \in \mathbb{H}^{-1}(\omega) \times \mathbb{H}^{-2}(\omega)$$

is a closed subspace of the space $\mathbb{H}^{-1}(\omega) \times \mathbb{H}^{-2}(\omega)$.

(b) *The following* Green's formula with little regularity *holds for all* $\boldsymbol{\eta} \in \boldsymbol{L}^2(\omega)$ *and all* $(\boldsymbol{n}, \boldsymbol{m}) \in \mathbb{H}_0^1(\omega) \times \mathbb{H}_0^2(\omega)$:

$$ {}_{\mathbb{H}^{-1}(\omega)\times\mathbb{H}^{-2}(\omega)}\big\langle \big(\boldsymbol{\gamma}(\boldsymbol{\eta}), \boldsymbol{\rho}(\boldsymbol{n})\big), (\boldsymbol{n}, \boldsymbol{m})\big\rangle_{\mathbb{H}_0^1(\omega)\times\mathbb{H}_0^2(\omega)} + \int_\omega \eta_i d^i(\boldsymbol{n}, \boldsymbol{m})\sqrt{a}\,\mathrm{d}y = 0, \tag{7} $$

where the functions $d^i(\boldsymbol{n}, \boldsymbol{m}) \in L^2(\omega)$ *are those defined in* (4).

(c) *Let the space* $\boldsymbol{L}^2(\omega)$ *be equipped with the inner product defined by*:

$$ \int_\omega \eta_i \zeta^i \sqrt{a} \quad \textit{for any } \boldsymbol{\eta} = (\eta_i) \in \boldsymbol{L}^2(\omega) \textit{ and } \boldsymbol{\zeta} = \big(\zeta^i\big) \in \boldsymbol{L}^2(\omega). $$

Then the dual operator of $(\boldsymbol{\gamma}, \boldsymbol{\rho}) : \boldsymbol{L}^2(\omega) \to \mathbb{H}^{-1}(\omega) \times \mathbb{H}^{-2}(\omega)$ *is*:

$$ (\boldsymbol{n}, \boldsymbol{m}) \in \mathbb{H}_0^1(\omega) \times \mathbb{H}_0^2(\omega) \to -\big(\sqrt{a}d^\alpha(\boldsymbol{n}, \boldsymbol{m}), \sqrt{a}d^3(\boldsymbol{n}, \boldsymbol{m})\big) \in \boldsymbol{L}^2(\omega). $$

Proof. The proof of (a) relies on the following *weak form of the Korn inequality on a surface* due to Ciarlet and Mardare [8]: There exists a constant C such that:

$$ \|\boldsymbol{\eta}\|_{\boldsymbol{L}^2(\omega)} < C\big(\|\boldsymbol{\eta}\|_{\boldsymbol{H}^{-1}(\omega)} + \big\|\boldsymbol{\gamma}(\boldsymbol{\eta})\big\|_{\mathbb{H}^{-1}(\omega)} + \big\|\boldsymbol{\rho}(\boldsymbol{\eta})\big\|_{\mathbb{H}^{-2}(\omega)}\big) \quad \text{for all } \boldsymbol{\eta} \in \boldsymbol{L}^2(\omega). $$

Note that, like in the proof of Theorem 3.1(b), the proof of this inequality again essentially hinges on the fundamental *lemma of J.-L. Lions*.

This inequality in turn implies that *there exists a constant* $\dot{C}$ *such that*:

$$ \|\dot{\boldsymbol{\eta}}\|_{\dot{\boldsymbol{L}}^2(\omega)} := \inf_{\boldsymbol{\xi}\in\mathbf{Ker}(\boldsymbol{\gamma},\boldsymbol{\eta})} \|\boldsymbol{\eta} + \boldsymbol{\xi}\|_{\boldsymbol{L}^2(\omega)} \leqslant \dot{C}\big(\big\|\boldsymbol{\gamma}(\dot{\boldsymbol{\eta}})\big\|_{\mathbb{H}^{-1}(\omega)} + \big\|\boldsymbol{\rho}(\dot{\boldsymbol{\eta}})\big\|_{\mathbb{H}^{-2}(\omega)}\big) $$

for all $\dot{\boldsymbol{\eta}} \in \dot{\boldsymbol{L}}^2(\Omega) = \boldsymbol{L}^2(\omega)/\mathbf{Ker}(\boldsymbol{\gamma}, \boldsymbol{\eta})$. Consequently, the operator $(\boldsymbol{\gamma}, \boldsymbol{\eta}) : \dot{\boldsymbol{L}}^2(\omega) \to \mathbb{H}^{-1}(\omega) \times \mathbb{H}^{-2}(\omega)$ is injective, continuous, and has an inverse from $\mathbf{Im}(\boldsymbol{\gamma}, \boldsymbol{\eta}) \subset \mathbb{H}^{-1}(\omega) \times \mathbb{H}^{-2}(\omega)$ onto $\dot{\boldsymbol{L}}^2(\omega)$ that is also continuous. Consequently, the space $\mathbf{Im}(\boldsymbol{\gamma}, \boldsymbol{\eta})$ is also a Banach space and thus a *closed* subspace of $\mathbb{H}^{-1}(\omega) \times \mathbb{H}^{-2}(\omega)$. The same conclusion clearly applies to $(\boldsymbol{\gamma}, \boldsymbol{\eta})$ now viewed as an operator acting from $\boldsymbol{L}^2(\omega)$ into $\mathbb{H}^{-1}(\omega) \times \mathbb{H}^{-2}(\omega)$, since the space $\mathbf{Im}(\boldsymbol{\gamma}, \boldsymbol{\eta})$ is the same in both cases. This proves (a).

The Green's formula (7) is established like the fundamental Green's formula (3), but with the integrals over ω appearing in its left-hand side now replaced with appropriate duality brackets. Note that there are no duality brackets on the boundary γ because $(\boldsymbol{n}, \boldsymbol{m}) \in \mathbb{H}_0^1(\omega) \times \mathbb{H}_0^2(\omega)$.

Part (c) is an immediate consequence of the Green's formula (7). □

4. Application to Donati compatibility conditions on a surface

The classical "*three-dimensional*" *Donati compatibility conditions* constitute a characterization of symmetric 3×3 matrix fields defined over a domain $\Omega \subset \mathbb{R}^3$ as linearized strain tensor fields. They take various forms, according to which boundary conditions are to be satisfied by the corresponding displacement vector field. A typical result in this direction is the following one (see Geymonat and Suquet [11], Geymonat and Krasucki [10], or Amrouche, Ciarlet, Gratie, and Kesavan [1]): Given a symmetric matrix field $\boldsymbol{e} = (e_{ij}) \in \mathbb{L}^2(\Omega)$, there exists a displacement vector field $\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega)$ such that $\boldsymbol{e} = \nabla_s \boldsymbol{v}$ in Ω if, and only if, $\int_\Omega e_{ij}s_{ij}\,\mathrm{d}x = 0$ for all matrix fields $(s_{ij}) \in \mathbb{L}^2(\Omega)$ that satisfy $\partial_j s_{ij} = 0$ in Ω.

The main objective of this Note and of the Note [7] is to extend this kind of compatibility conditions to *surfaces*, the linearized change of metric and change of curvature tensor fields $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})$ "replacing" the symmetrized gradient matrix field $\nabla_s \boldsymbol{v}$.

To this end, there are two different approaches. The first one, which is developed in this Note, is essentially based on the *Green's formulas with little regularity on a surface* of Theorems 3.1 and 3.2, and on *Banach closed range theorem*. The other approach, which will be developed in the Note [7], is essentially based on various properties of the "surface analogue" of the classical space $\boldsymbol{H}(\mathrm{div}, \cdot)$.

To begin with, we obtain *Donati compatibility conditions* that are necessary and sufficient for recovering from its linearized change of metric and change of curvature tensors a displacement field that satisfies a *homogeneous Dirichlet boundary condition on the entire boundary*.

Theorem 4.1. *Let there be given a domain* ω *in* $\mathbb{R}^2$ *with a boundary of class* $\mathcal{C}^{1,1}$, *an immersion* $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega})$, *and two tensor fields* $\boldsymbol{c} = (c_{\alpha\beta}) \in \mathbb{L}^2(\omega)$ *and* $\boldsymbol{r} = (r_{\alpha\beta}) \in \mathbb{L}^2(\omega)$. *Then there exists a vector field* $\boldsymbol{\zeta} = ((\zeta_\alpha), \zeta_3) \in \boldsymbol{H}_0^1(\omega) \times H_0^2(\omega)$ *such that*

$$ \gamma_{\alpha\beta}(\boldsymbol{\zeta}) = c_{\alpha\beta} \quad \textit{and} \quad \rho_{\alpha\beta}(\boldsymbol{\zeta}) = r_{\alpha\beta} \quad \textit{in } L^2(\omega), $$

if and only if

$$\int_\omega \left(n^{\alpha\beta} c_{\alpha\beta} + m^{\alpha\beta} r_{\alpha\beta}\right)\sqrt{a}\, \mathrm{d}y = 0$$

for all $(\boldsymbol{n}, \boldsymbol{m}) = ((n^{\alpha\beta}), (m^{\alpha\beta})) \in \mathbb{L}^2(\omega) \times \mathbb{L}^2(\omega)$ *that satisfy* (*the distributions* $d^i(\boldsymbol{n}, \boldsymbol{m})$ *are defined in* (4))

$$d^\alpha(\boldsymbol{n}, \boldsymbol{m}) = 0 \quad \text{in } H^{-1}(\omega) \quad \text{and} \quad d^3(\boldsymbol{n}, \boldsymbol{m}) = 0 \quad \text{in } H^{-2}(\omega).$$

If this is the case, such a vector field $\boldsymbol{\zeta} \in \boldsymbol{H}_0^1(\omega) \times \boldsymbol{H}_0^2(\omega)$ *is unique.*

Sketch of proof. The assertion follows from parts (a) and (c) of Theorem 3.1, combined with *Banach closed range theorem*. That $\mathbf{Ker}(\boldsymbol{\gamma}, \boldsymbol{\rho}) = \{\mathbf{0}\}$ in this case implies that the vector field $\boldsymbol{\zeta}$ is unique. □

We next obtain *Donati compatibility conditions* that are necessary and sufficient for recovering from its linearized change of metric and change of curvature tensors a displacement field that satisfies a *homogeneous Neumann boundary condition on the entire boundary*.

Theorem 4.2. *Let there be given a domain* ω *in* $\mathbb{R}^2$ *with a boundary of class* $\mathcal{C}^{1,1}$, *an immersion* $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega})$, *and two tensor fields* $\boldsymbol{c} = (c_{\alpha\beta}) \in \mathbb{L}^2(\omega)$ *and* $\boldsymbol{r} = (r_{\alpha\beta}) \in \mathbb{L}^2(\omega)$. *Then there exists a vector field* $\boldsymbol{\zeta} = ((\zeta_\alpha), \zeta_3) \in \boldsymbol{H}^1(\omega) \times H^2(\omega)$ *such that*:

$$\gamma_{\alpha\beta}(\boldsymbol{\zeta}) = c_{\alpha\beta} \quad \text{and} \quad \rho_{\alpha\beta}(\boldsymbol{\zeta}) = r_{\alpha\beta} \quad \text{in } L^2(\omega),$$

if and only if

$$\int_\omega \left(n^{\alpha\beta} c_{\alpha\beta} + m^{\alpha\beta} r_{\alpha\beta}\right)\sqrt{a}\, \mathrm{d}y = 0$$

for all $(\boldsymbol{n}, \boldsymbol{m}) = ((n^{\alpha\beta}), (m^{\alpha\beta})) \in \mathbb{H}_0^1(\omega) \times \mathbb{H}_0^2(\omega)$ *that satisfy* (*cf.* (4))

$$d^i(\boldsymbol{n}, \boldsymbol{m}) = 0 \quad \text{in } L^2(\omega).$$

If this is the case, all other vector fields $\tilde{\boldsymbol{\zeta}} = (\tilde{\zeta}_i) \in \boldsymbol{H}^1(\omega) \times H^2(\omega)$ *satisfying* $\boldsymbol{\gamma}(\tilde{\boldsymbol{\zeta}}) = \boldsymbol{c}$ *and* $\rho(\tilde{\boldsymbol{\zeta}}) = \boldsymbol{r}$ *in* $\mathbb{L}^2(\omega)$ *are such that*

$$\tilde{\zeta}_i(y)\boldsymbol{a}^i(y) = \zeta_i(y)\boldsymbol{a}^i(y) + \boldsymbol{a} + \boldsymbol{b} \wedge \boldsymbol{\theta}(y) \quad \text{for almost all } y \in \omega,$$

for some vectors $\boldsymbol{a} \in \mathbb{R}^3$ *and* $\boldsymbol{b} \in \mathbb{R}^3$.

Sketch of proof. Since the dual of the operator $(\boldsymbol{\gamma}, \boldsymbol{\rho}) : \boldsymbol{L}^2(\omega) \to \mathbb{H}^{-1}(\omega) \times \mathbb{H}^{-2}(\omega)$ is the operator:

$$-\sqrt{a}\boldsymbol{d} : \mathbb{H}_0^1(\omega) \times \mathbb{H}_0^2(\omega) \to \boldsymbol{L}^2(\omega) \quad \text{where } \boldsymbol{d} = \left(d^i\right),$$

and since the image $\mathbf{Im}(\boldsymbol{\gamma}, \boldsymbol{\rho})$ of this operator $(\boldsymbol{\gamma}, \boldsymbol{\rho})$ is closed in $\mathbb{H}^{-1}(\omega) \times \mathbb{H}^{-2}(\omega)$ (Theorem 3.2), *Banach closed range theorem* asserts that:

$$\mathbf{Im}(\boldsymbol{\gamma}, \boldsymbol{\rho}) = \Big\{(\boldsymbol{c}, \boldsymbol{r}) \in \mathbb{H}^{-1}(\omega) \times \mathbb{H}^{-2}(\omega); \\ {}_{\mathbb{H}^{-1}(\omega)\times\mathbb{H}^{-2}(\omega)}\big\langle(\boldsymbol{c}, \boldsymbol{r}), (\boldsymbol{n}, \boldsymbol{m})\big\rangle_{\mathbb{H}_0^1(\omega)\times\mathbb{H}_0^2(\omega)} = 0 \text{ for all } (\boldsymbol{n}, \boldsymbol{m}) \in \mathbf{Ker}(\sqrt{a}\boldsymbol{d})\Big\}.$$

Let then tensor fields $\boldsymbol{c} = (c_{\alpha\beta}) \in \mathbb{L}^2(\omega) \subset \mathbb{H}^{-1}(\omega)$ and $\boldsymbol{r} = (r_{\alpha\beta}) \in \mathbb{L}^2(\omega) \subset \mathbb{H}^{-2}(\omega)$ be such that:

$$\int_\omega \left(n^{\alpha\beta} c_{\alpha\beta} + m^{\alpha\beta} r_{\alpha\beta}\right)\sqrt{a}\, \mathrm{d}y = {}_{\mathbb{H}^{-1}(\omega)\times\mathbb{H}^{-2}(\omega)}\big\langle(\boldsymbol{c}, \boldsymbol{r}), (\boldsymbol{n}, \boldsymbol{m})\big\rangle_{\mathbb{H}_0^1(\omega)\times\mathbb{H}_0^2(\omega)} = 0$$

for all $(\boldsymbol{n}, \boldsymbol{m}) = ((n^{\alpha\beta}), (m^{\alpha\beta})) \in \mathbb{H}_0^1(\omega) \times \mathbb{H}_0^2(\omega)$ that satisfy $d^i(\boldsymbol{n}, \boldsymbol{m}) = 0$ in $L^2(\omega)$. Hence there exists a vector field $\boldsymbol{\zeta} = (\zeta_i) \in \boldsymbol{L}^2(\omega)$ such that $\gamma_{\alpha\beta}(\boldsymbol{\zeta}) = c_{\alpha\beta}$ in $H^{-1}(\omega)$ and $\rho_{\alpha\beta}(\boldsymbol{\zeta}) = r_{\alpha\beta}$ in $H^{-2}(\omega)$. One then shows, using J.-L. Lions' lemma, that $\zeta_\alpha \in H^1(\omega)$ and $\zeta_3 \in H^2(\omega)$.

That all other solutions $\tilde{\boldsymbol{\zeta}}$ are of the form indicated in the statement of the theorem follows from a "weak" version of the *infinitesimal rigid displacement lemma on a surface* established in [8]. □

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