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Marcinkiewicz r-classes and Fourier series expansions of operator ergodic Stieltjes convolutions

$\mathfrak{M}_r(\mathbb{T})$, *séries d'opérateurs de Fourier et « Stieltjes convolutions »*

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A B S T R A C T

We study the Fourier series expansions in the strong operator topology for operator-valued Stieltjes convolutions of Marcinkiewicz r-classes against spectral decompositions of modulus-mean-bounded operators. The vector-valued harmonic analysis resulting can be viewed as an extension of traditional Calderón–Coifman–G. Weiss transference without being constrained by the latter's requirement of power-boundedness.

R É S U M É

Cette note étudie (dans la topologie forte des opérateurs) les développements en séries de Fourier pour les « convolutions de Stieltjes » des fonctions dans les r-classes de Marcinkiewicz par $\mathrm{d}E$, où E est la décomposition spectrale d'une bijection linéaire arbitraire T telle que T soit un opérateur préservant la disjonction dont le module linéaire est à moyennes bornées. L'analyse harmonique vectorielle qui en résulte étend le transfert traditionnel de Calderón–Coifman–G. Weiss, sans supposer les puissances uniformément bornées traditionnellement requises pour le transfert.

1. Introduction

For $1 < p < \infty$, and μ a σ-finite measure, we consider an arbitrary disjoint invertible operator $T \in \mathfrak{B}(L^p(\mu))$ such that T *is modulus-mean-bounded* (abbreviated mmb)—that is its linear modulus $|T|$ satisfies $\sup_{n\geqslant 0} \|(2n+1)^{-1}\sum_{j=-n}^{n} |T|^j\| < \infty$. It is well known [3] that the mmb operator T is automatically a *trigonometrically well-bounded* (abbreviated twb) operator. In brief, this means that T has a "unitary-like" spectral representation $T = \int_{0^-}^{2\pi} e^{i\lambda}\, \mathrm{d}E(\lambda)$, where $E : \mathbb{R} \to \mathfrak{B}(L^p(\mu))$ is a certain unique idempotent-valued function (the spectral decomposition of T, henceforth denoted E) having properties similar to, but weaker than, those that would be inherited from a Borel spectral measure, and the Stieltjes integral on the right is taken with respect to the strong operator topology. (More precise background information regarding this "spectral integration" will be given shortly, but for a detailed review thereof, as well as of the features of the r-variation of functions discussed below, see [2].) This note develops the Fourier series expansions (in the strong operator topology) for operator ergodic Stieltjes convolutions of Marcinkiewicz r-functions [5] with $\mathrm{d}E$. The present study can be viewed as furnishing a mechanism whereby the mmb operator T transfers the classical Fourier series expansions associated with $\mathfrak{M}_r(\mathbb{T})$ classes to $\mathfrak{B}(L^p(\mu))$-valued

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Fourier series expansions, thereby extending the scope of traditional Calderón–Coifman–G. Weiss transference methodology without tying T to the latter's requirement that the transference vehicle be power-bounded.

The notion of *spectral family of projections* $\mathcal{E}$ in a Banach space $\mathfrak{X}$ is at the heart of these considerations. By definition $\mathcal{E}(\cdot):\mathbb{R}\to\mathfrak{B}(\mathfrak{X})$ is an idempotent-valued function such that: (a) $\mathcal{E}(\lambda)\mathcal{E}(\tau)=\mathcal{E}(\tau)\mathcal{E}(\lambda)=\mathcal{E}(\lambda)$ if $\lambda\leqslant\tau$; (b) $\|\mathcal{E}\|_u\equiv\sup\{\|\mathcal{E}(\lambda)\|:\ \lambda\in\mathbb{R}\}<\infty$; (c) with respect to the strong operator topology, $\mathcal{E}(\cdot)$ is right-continuous and has a left-hand limit $\mathcal{E}(\lambda^-)$ at each point $\lambda\in\mathbb{R}$; (d) with respect to the strong operator topology, $\mathcal{E}(\lambda)\to I$ as $\lambda\to\infty$ and $\mathcal{E}(\lambda)\to 0$ as $\lambda\to-\infty$. (*If, in addition, there exist* $a,b\in\mathbb{R}$ *with* $a\leqslant b$ *such that* $\mathcal{E}(\lambda)=0$ *for* $\lambda<a$ *and* $\mathcal{E}(\lambda)=I$ *for* $\lambda\geqslant b$, $\mathcal{E}(\cdot)$ *is said to be concentrated on* $[a,b]$.) When $\mathcal{E}(\cdot)$ is concentrated on a compact interval $J=[a,b]$, an associated Stieltjes-like notion of *spectral integration* against $\mathcal{E}$ can be formulated (relative to the strong operator topology) for suitable bounded complex-valued functions defined on J. When the spectral integral of a function ψ exists (written $\int_J\psi(\lambda)\,\mathrm{d}\mathcal{E}(\lambda)$), we put $\int_J^{\oplus}\psi(\lambda)\,\mathrm{d}\mathcal{E}(\lambda)\equiv\psi(a)\mathcal{E}(a)+\int_J\psi(\lambda)\,\mathrm{d}\mathcal{E}(\lambda)$. For all functions f having bounded variation on J, the spectral integral $\int_J f\,\mathrm{d}\mathcal{E}$ automatically exists. The definition of twb operator can now be stated precisely as follows. An operator $U\in\mathfrak{B}(\mathfrak{X})$ is *twb* if there is a spectral family $\mathcal{E}(\cdot)$ in $\mathfrak{X}$ concentrated on $[0,2\pi]$ such that $U=\int_{[0,2\pi]}^{\oplus}\mathrm{e}^{\mathrm{i}\lambda}\,\mathrm{d}\mathcal{E}(\lambda)$. In this case, it is possible to arrange that $\mathcal{E}(2\pi^-)=I$, and then the spectral family $\mathcal{E}(\cdot)$ is uniquely determined by U, and called *the spectral decomposition* of U. We denote by $\mathcal{E}_z$ the spectral decomposition of zU. It bears mention here that T is inextricably bound up with its operator ergodic Hilbert transform, in view of the following recent characterization of twb operators [1]: if $\mathfrak{X}$ is a super-reflexive Banach space, and $U\in\mathfrak{B}(\mathfrak{X})$ is invertible, then U is twb if and only if $\sup\{\|\sum_{0<|k|\leqslant n}\frac{z^k}{k}U^k\|:\ n\in\mathbb{N},\ z\in\mathbb{T}\}<\infty$.

Let $\mathrm{var}_r(f,\mathbb{T})\equiv\sup\{\sum_{k=1}^{N}|f(\mathrm{e}^{\mathrm{i}x_k})-f(\mathrm{e}^{\mathrm{i}x_{k-1}})|^r\}^{1/r}$, for $1\leqslant r<\infty$ and $f:\mathbb{T}\to\mathbb{C}$, where the supremum is taken over all partitions $0=x_0<x_1<\cdots<x_N=2\pi$. The Banach algebra $V_r(\mathbb{T})$ consists of all f for which $\mathrm{var}_r(f,\mathbb{T})<\infty$, and has norm specified by $\|f\|_{V_r(\mathbb{T})}\equiv\sup_{z\in\mathbb{T}}|f(z)|+\mathrm{var}_r(f,\mathbb{T})$. (Analogous definitions apply for a compact interval $[a,b]$ in place of $\mathbb{T}$.) The Fourier series of any $f\in V_r(\mathbb{T})$ converges pointwise on $\mathbb{T}$. The *dyadic points* relevant to the study of periodic functions are the terms of the sequence $\{t_k\}_{k=-\infty}^{\infty}\subseteq(0,2\pi)$ given by $t_k=2^{k-1}\pi$ (respectively, $2\pi-2^{-k}\pi$), if $k\leqslant 0$ (respectively $k>0$). With this notation, the *Marcinkiewicz r-class* $\mathfrak{M}_r(\mathbb{T})$, $1\leqslant r<\infty$, is the Banach algebra of all functions $\phi:\mathbb{T}\to\mathbb{C}$ such that $\|\phi\|_{\mathfrak{M}_r(\mathbb{T})}\equiv\sup_{z\in\mathbb{T}}|\phi(z)|+\sup_{k\in\mathbb{Z}}\mathrm{var}_r(\phi(\mathrm{e}^{\mathrm{i}(\cdot)}),[t_k,t_{k+1}])<\infty$. The classes $\mathfrak{M}_r(\mathbb{T})$ consist of Fourier multipliers for various weighted and unweighted settings. While the classes $\mathfrak{M}_r(\mathbb{T})$ are broader than the classes $V_r(\mathbb{T})$, they are less tractable. In particular, the convergence situation for $\phi\in\mathfrak{M}_r(\mathbb{T})$ at $z=1$ is a more delicate matter—on the pleasant side, individual weaker conditions than continuity of $\phi\in\mathfrak{M}_r(\mathbb{T})$ at $z=1$ suffice. We also comment that the context of $T\in\mathfrak{B}(L^p(\mu))$ can be illustrated by specializing T to be the left bilateral shift $\mathcal{L}\in\mathfrak{B}(\ell^p(w))$, where $w\equiv\{w_k\}_{k=-\infty}^{\infty}$ is a positive weight sequence satisfying the discrete Muckenhoupt A_p weight condition on $\mathbb{Z}$. In particular, for appropriate choices of w, we can have $\sup_{n\in\mathbb{Z}}\|\mathcal{L}^n\|=\infty$. In the setting of $\mathcal{L}$ our results can be expressed in terms of Fourier multiplier theory for weighted Lebesgue spaces. For more background details on this specialization, see, e.g., Sections 3–5 of [3] and the method of proof for Theorem 4.3 in [4]. In all that follows, the symbol "K" with a (possibly empty) set of subscripts will signify a constant which depends only on those subscripts, and which may change in value from one occurrence to another.

2. Fourier series for operator-valued "Stieltjes convolutions"

Our goal is to expand the following result (see Theorem 4.1 of [2]) from $V_r(\mathbb{T})$ to $\mathfrak{M}_r(\mathbb{T})$ by specializing its twb operator U to T.

Theorem 2.1. *Let $\mathfrak{X}$ be a super-reflexive Banach space, and let $U\in\mathfrak{B}(\mathfrak{X})$ be twb, with spectral decomposition $\mathcal{E}$. Then there is $q_1\equiv q_1(U)\in(1,\infty)$ such that whenever $r\in[1,q_1(U))$ and $\psi\in V_r(\mathbb{T})$, we can use spectral integration to define the function $\Psi_U:\mathbb{T}\to\mathfrak{B}(\mathfrak{X})$ specified by $\Psi_U(z)\equiv\int_{[0,2\pi]}^{\oplus}\psi(z\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}\mathcal{E}(t)$, which has the following properties: for each $x\in\mathfrak{X}$, the series $\sum_{\nu=-\infty}^{\infty}\widehat{\psi}(\nu)z^{\nu}U^{\nu}x$ $(z\in\mathbb{T})$ is the Fourier series of $\Psi_U(\cdot)x:\mathbb{T}\to\mathfrak{X}$, and converges in the norm topology at each $z\in\mathbb{T}$; $\sup_{z\in\mathbb{T}}\|\Psi_U(z)\|\leqslant K_{r,U}\|\psi\|_{V_r(\mathbb{T})}$.*

The classes $V_r(\mathbb{T})$ occurring in the framework of Theorem 2.1 cannot be replaced there by $\mathfrak{M}_1(\mathbb{T})$, since it is known that there are a trigonometrically well-bounded operator U_0 acting on the Hilbert sequence space $\mathfrak{X}=\ell^2(\mathbb{N})$ and a function belonging to $\mathfrak{M}_1(\mathbb{T})$ that cannot be integrated against the spectral decomposition of U_0 (see the proof of Theorem 6.1 in [4])—moreover, in contrast to $V_r(\mathbb{T},)$ the class $\mathfrak{M}_r(\mathbb{T})$ is not rotation invariant. However, in the setting of the mmb operator T the following theorem (see pages 29–30 of [2]) furnishes the requisite spectral integration against E of rotated $\mathfrak{M}_r(\mathbb{T})$ classes for formulating a counterpart of Theorem 2.1. In what follows, we put $\gamma(T)=\sup\{\|\frac{1}{2N+1}\sum_{n=-N}^{N}|T|^n\|:\ N\geqslant 0\}<\infty$. We remark that $\gamma(T)=\gamma(T^{-1})=\gamma(zT)=\gamma(|T|)$, for all $z\in\mathbb{T}$, and also that the parameter $q_1(T)$ furnished by Theorem 2.1 can and henceforth will be chosen to satisfy $q_1(T)=q_1(T^{-1})=q_1(zT)$, for all $z\in\mathbb{T}$. These two observations permit us to substitute zT or T^{-1} for T in various arguments without loss of generality.

Theorem 2.2. *There is $q_2=q_2(p,\gamma(T))\in(1,\infty)$ such that for every $r\in[1,q_2)$, each $\phi\in\mathfrak{M}_r(\mathbb{T})$, and every $z\in\mathbb{T}$, the spectral integral $\Phi_T(z)\equiv\int_{[0,2\pi]}^{\oplus}\phi(z\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}E(t)$ exists and equals $\int_{[0,2\pi]}^{\oplus}\phi(\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}E_z(t)$. $\|\int_{[0,2\pi]}^{\oplus}\phi(z\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}E(t)\|\leqslant K_{p,r,\gamma(T)}\|\phi\|_{\mathfrak{M}_r(\mathbb{T})}$.*

We can now state our central result as follows.

Theorem 2.3. *Let $q_3(T) \in (1,\infty)$ be the minimum of $q_2(p,\gamma(T)) \in (1,\infty)$ and $q_1(T) \in (1,\infty)$. Suppose that $r \in [1,q_3)$, $\psi \in \mathfrak{M}_r(\mathbb{T})$ is a continuous function on $\mathbb{T}$, and $f \in L^p(\mu)$. Then for each $k \in \mathbb{Z}$, the kth Fourier coefficient of the bounded vector-valued function $\Psi_T(\cdot)f : \mathbb{T} \mapsto L^p(\mu)$ is expressed by $(\Psi_T(\cdot)f)^\wedge(k) = \widehat{\psi}(k)T^k f$. The function $\Psi_T(\cdot)f$ is continuous on $\mathbb{T}$ in the norm topology of $L^p(\mu)$, and hence the $(C,1)$-means of its Fourier series converge to $\Psi_T(\cdot)f$ uniformly in $z \in \mathbb{T}$, with respect to the norm topology of $L^p(\mu)$.*

The following lemma can be established by a reduction to the setting of functions of bounded r-variation, which is then covered by the abstract Theorem 3.11 of [1].

Lemma 2.4. *Suppose that $r \in [1,q_3)$, and $\{\phi_\gamma\}_{\gamma\in\Gamma} \subseteq \mathfrak{M}_r(\mathbb{T})$ is a net such that: for each $\gamma \in \Gamma$, the function $t \in (0,2\pi) \mapsto \phi_\gamma(e^{it})$ is left-continuous on $(0,2\pi)$; the net $\{\phi_\gamma\}_{\gamma\in\Gamma}$ converges pointwise on $\mathbb{T}$ to a function $\phi : \mathbb{T} \to \mathbb{C}$; $\sup\{\|\phi_\gamma\|_{\mathfrak{M}_r(\mathbb{T})}\colon \gamma \in \Gamma\} < \infty$. Then $\phi \in \mathfrak{M}_r(\mathbb{T})$, and $\{\int_{[0,2\pi]}^{\oplus} \phi_\gamma(e^{it})\,dE(t)\}_{\gamma\in\Gamma}$ converges in the strong operator topology of $\mathfrak{B}(L^p(\mu))$ to $\int_{[0,2\pi]}^{\oplus} \phi(e^{it})\,dE(t)$.*

Sketch of proof for Theorem 2.3. For $n \in \mathbb{N}$ let χ_n be the characteristic function on $\mathbb{T}$ of $\{e^{it}\colon t \in (t_{-n},t_n]\} \cup \{1\}$, and define $\psi_n \in V_r(\mathbb{T})$ to be $\psi\chi_n$. Since $\|\psi_n\|_{\mathfrak{M}_r(\mathbb{T})} \leqslant K\|\psi\|_{\mathfrak{M}_r(\mathbb{T})}$, we see from Lemma 2.4 that for each $z \in \mathbb{T}$, $\{\int_{[0,2\pi]}^{\oplus} \psi_n(e^{it})\,dE_z(t)\}_{n=1}^{\infty}$ converges to $\int_{[0,2\pi]}^{\oplus} \psi(e^{it})\,dE_z(t) = \int_{[0,2\pi]}^{\oplus} \psi(ze^{it})\,dE(t)$. Application of Theorem 2.1 shows that for $k \in \mathbb{Z}$, $((\Psi_n)_T f)^\wedge(k) = \widehat{\psi}_n(k)T^k f$, and recourse to Dominated Convergence now gives $(\Psi_T(\cdot)f)^\wedge(k) = \widehat{\psi}(k)T^k f$. For global continuity it suffices to show that $\Psi_T(\cdot)f$ is continuous at $z=1$, since this outcome still at $z=1$ would then automatically hold for (wT) and $E_w(\cdot)$ in place of T and $E(\cdot)$, while $\int_{[0,2\pi]}^{\oplus} \psi(ze^{it})\,dE_w(t) = \int_{[0,2\pi]}^{\oplus} \psi(wze^{it})\,dE(t)$. This reduces to showing that for fixed $n \in \mathbb{N}$, we have, in the strong operator topology of $\mathfrak{B}(L^p(\mu))$, $\lim_{\theta\to 0} \int_{[0,2\pi]}^{\oplus} \eta_n(e^{it})\psi(e^{i\theta}e^{it})\,dE(t) = \int_{[0,2\pi]}^{\oplus} \eta_n(e^{it})\psi(e^{it})\,dE(t)$, where η_n is the characteristic function of $\{e^{it}\colon t \in (t_{-n},t_n]\}$. For $|\theta| < 2^{-n-2}\pi$,

$$\operatorname{var}_r\big(\eta_n\big(e^{i(\cdot)}\big)\psi\big(e^{i\theta}e^{i(\cdot)}\big),[0,2\pi]\big) \leqslant \operatorname{var}_r\big(\psi\big(e^{i(\cdot)}\big),[t_{-n-1},t_{n+1}]\big) + 2\|\psi\|_\infty,$$

whence $\sup\{\|\eta_n(e^{i(\cdot)})\psi(e^{i\theta}e^{i(\cdot)})\|_{V_r([0,2\pi])}\colon |\theta| < 2^{-n-2}\pi\} < \infty$. This completes the proof upon appeal to Lemma 2.4. □

It is an open question whether the pointwise Fourier series $(C,1)$-summability in the strong operator topology that Theorem 2.3 accords the setting of $\mathfrak{M}_r(\mathbb{T})$-classes can be improved to the pointwise convergence that Theorem 2.1 associates with $V_r(\mathbb{T})$-classes. However, these two theorems can act in concert with suitably devised localization techniques for the Fourier series involved to yield the following result in the positive direction.

Theorem 2.5. *Under the hypotheses of Theorem 2.3, and with all convergence in the strong operator topology, $\sum_{k=-\infty}^{\infty} \widehat{\psi}(k) \cdot T^k\{E(2\pi-\delta) - E(\delta)\}$ converges to $\Psi_T(1)\{E(2\pi-\delta) - E(\delta)\}$ for each $\delta \in (0,\pi)$, and hence as $\delta \to 0$ this series approaches $\Psi_T(1) - \psi(1)E(0)$.*

References

[1] E. Berkson, Spectral theory and operator ergodic theory on super-reflexive Banach spaces, Stud. Math. 200 (2010) 221–246.
[2] E. Berkson, Rotation methods in operator ergodic theory, N.Y. J. Math. 17 (2011) 21–39, located online at http://nyjm.albany.edu/j/2011/17-2v.pdf.
[3] E. Berkson, T.A. Gillespie, Mean-boundedness and Littlewood–Paley for separation-preserving operators, Trans. Amer. Math. Soc. 349 (1997) 1169–1189.
[4] E. Berkson, T.A. Gillespie, Shifts as models for spectral decomposability on Hilbert space, J. Oper. Theory 50 (2003) 77–106.
[5] R. Coifman, J.L. Rubio de Francia, S. Semmes, Multiplicateurs de Fourier de $L^p(\mathbb{R})$ et estimations quadratiques, C. R. Acad. Sci. Paris, Ser. I 306 (1988) 351–354.