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A limiting weak type estimate for capacitary maximal function ☆

Une estimation de type faible limite pour la fonction maximale capacitaire

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ABSTRACT

A capacitary analogue of the limiting weak type estimate of P. Janakiraman for the Hardy–Littlewood maximal function of an $L^1(\mathbb{R}^n)$-function (cf. [5,6]) is discovered.

RÉSUMÉ

Pour l'analogue en termes de capacités de la fonction maximale de Hardy–Littlewood, on démontre une estimation de type faible limite correspondant à celle de P. Janakiraman.

1. Statement of theorem

For an L^1_{loc}-integrable function f on $\mathbb{R}^n$, $n \geqslant 1$, let $Mf(x) = \sup_{x\in B}(\mathscr{L}(B))^{-1}\int_B |f(y)|\,dy$ denote the Hardy–Littlewood maximal function of f at $x \in \mathbb{R}^n$, where the supremum is taken over all Euclidean balls B containing x and $\mathscr{L}(B)$ stands for the n-dimensional Lebesgue measure of B. Among several results of [5,6], P. Janakiraman obtained the following fundamental limit:

$$\lim_{\lambda\to 0} \lambda\mathscr{L}\big(\big\{x\in\mathbb{R}^n\colon\ Mf(x) > \lambda\big\}\big) = \|f\|_1 = \int_{\mathbb{R}^n}\big|f(y)\big|\,dy \quad \forall f \in L^1\big(\mathbb{R}^n\big).$$

This note studies the limiting weak-type estimate for a capacity. To be more precise, recall that a set function $C(\cdot)$ on $\mathbb{R}^n$ is said to be a capacity (cf. [2,3]) provided that:

$$\begin{cases} C(\emptyset) = 0; \\ 0 \leqslant C(A) \leqslant \infty \quad \forall A \subseteq \mathbb{R}^n; \\ C(A) \leqslant C(B) \quad \forall A \subseteq B \subseteq \mathbb{R}^n; \\ C\left(\bigcup_{i=1}^{\infty} A_i\right) \leqslant \sum_{i=1}^{\infty} C(A_i) \quad \forall A_i \subseteq \mathbb{R}^n. \end{cases}$$

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For a given capacity $C(\cdot)$ let:

$$M_C f(x) = \sup_{x\in B} \frac{1}{C(B)} \int_B |f(y)|\,dy$$

be the capacitary maximal function of an L^1_{loc}-integrable function f at x for which the supremum ranges over all Euclidean balls B containing x; see also [7].

In order to establish a capacitary analogue of the last limit formula for $f \in L^1(\mathbb{R}^n)$, we are required to make the following natural assumptions:

- Assumption 1 – the capacity $C(B(x,r))$ of the ball $B(x,r)$ centered at x with radius r is a function depending on r only, but also the capacity $C(\{x\})$ of the set $\{x\}$ of a single point $x \in \mathbb{R}^n$ equals 0.
- Assumption 2 – there are two nonnegative functions ϕ and ψ on $(0,\infty)$ such that:

$$\begin{cases} \phi(t)C(E) \leqslant C(tE) \leqslant \psi(t)C(E) \quad \forall t>0 \quad \text{and} \quad tE = \{tx \in \mathbb{R}^n:\ x \in E \subseteq \mathbb{R}^n\}; \\ \lim_{t\to 0}\phi(t) = 0 = \lim_{t\to 0}\psi(t) \quad \text{and} \quad \lim_{t\to 0}\psi(t)/\phi(t) = \tau \in (0,\infty). \end{cases}$$

Theorem 1.1. *Under the above-mentioned two assumptions, one has*:

$$\lim_{\lambda\to 0} \lambda C\big(\{x \in \mathbb{R}^n:\ M_C f(x) > \lambda\}\big) \approx \|f\|_1 \quad \forall f \in L^1(\mathbb{R}^n).$$

Here and henceforth, $\mathsf{X} \approx \mathsf{Y}$ *means that there is a constant* $c > 0$ *independent of* X *and* Y *such that* $c^{-1}\mathsf{Y} \leqslant \mathsf{X} \leqslant c\mathsf{Y}$.

Note that the $(0,n] \ni (n-\lambda)$-dimensional Hausdorff content $\Lambda^{(\infty)}_{n-\lambda}$ and the $1 < p$-variational capacity obey Assumptions 1–2 (cf. [1,9]). So, an application of Theorem 1.1 to $C = \Lambda^{(\infty)}_{n-\lambda}$ actually reveals that the real interpolation between $L^1(\mathbb{R}^n)$ and the Morrey space $\mathcal{L}^{1,\lambda}(\mathbb{R}^n)$ (of all functions f with $M_{\Lambda^{(\infty)}_{n-\lambda}} f \in L^\infty(\mathbb{R}^n)$):

$$\|f\|_{(L^1,\mathcal{L}^{1,\lambda})_{1-p^{-1},p}} \approx \|M_{\Lambda^{(\infty)}_{n-\lambda}} f\|_{L^p(\Lambda^{(\infty)}_{n-\lambda})} \approx \left(\int_0^\infty \Lambda^{(\infty)}_{n-\lambda}\big(\{x \in \mathbb{R}^n:\ M_{\Lambda^{(\infty)}_{n-\lambda}} f(x) > t\}\big)\,dt^p\right)^{\frac{1}{p}}$$

established in [8, Theorem 3] is a natural extension of the classical real interpolation $\|f\|_{(L^1,L^\infty)_{1-p^{-1},p}} \approx \|Mf\|_{L^p}$.

2. Four lemmas

To prove Theorem 1.1, we will always suppose that $C(\cdot)$ is a capacity obeying Assumptions 1–2 above, but also need four lemmas based on the following capacitary maximal function $M_C\nu$ of a finite nonnegative Borel measure ν on $\mathbb{R}^n$:

$$M_C\nu(x) = \sup_{B\ni x} \frac{\nu(B)}{C(B)} \quad \forall x \in \mathbb{R}^n,$$

where the supremum is taken over all balls $B \subseteq \mathbb{R}^n$ containing x.

Lemma 2.1. *If* δ_0 *is the delta measure at the origin, then* $\lambda C(\{x \in \mathbb{R}^n:\ M_C\delta_0(x) > \lambda\}) = 1$.

Proof. According to the definition of the delta measure and Assumptions 1–2, we have:

$$M_C\delta_0(x) = \frac{1}{C(B(x,|x|))} \quad \forall |x| \neq 0.$$

Now, if x obeys $M_C\delta_0(x) > \lambda$, then $\lambda C(B(x,|x|)) < 1$. Note that if $C(B(0,r))$ equals $\frac{1}{\lambda}$, then one has the following property:

$$\begin{cases} C\big(B(x,|x|)\big) < \frac{1}{\lambda} \quad \forall |x| < r; \\ C\big(B(x,|x|)\big) = \frac{1}{\lambda} \quad \forall |x| = r; \\ C\big(B(x,|x|)\big) > \frac{1}{\lambda} \quad \forall |x| > r. \end{cases}$$

Thus, $\{x \in \mathbb{R}^n:\ M_C\delta_0(x) > \lambda\} = B(0,r)$, and consequently, $C(\{x \in \mathbb{R}^n:\ M_C\delta_0(x) > \lambda\}) = C(B(0,r)) = \lambda^{-1}$. □

Lemma 2.2. *If ν is a finite nonnegative Borel measure on $\mathbb{R}^n$ with $\nu(\mathbb{R}^n)=1$, then $\lambda\lim_{t\to 0}C(\{x\in\mathbb{R}^n:\ M_C\nu_t(x)>\lambda\})=1$, where $t>0$; $\nu_t(E)=\nu(\frac{1}{t}E)$; $\frac{1}{t}E=\{\frac{x}{t}:\ x\in E\}$; $E\subseteq\mathbb{R}^n$.*

Proof. For two positive numbers ϵ and η, choose ϵ_1 small relative to both ϵ and η, but also let t be small and the induced ϵ_t be such that: $\nu_t(B(0,\epsilon_t))>1-\epsilon$; $\epsilon_t=3^{-1}\epsilon_1$; $\lim_{t\to 0}\epsilon_t=0$; $\epsilon<\eta C(B(0,\epsilon_1))$. Now, if:

$$\begin{cases} E^t_{1,\lambda}=\left\{x\in\mathbb{R}^n\setminus B(0,\epsilon_1):\ \lambda<M_C\nu_t(x)\leqslant\dfrac{1}{C(B(x,|x|-\epsilon_t))}\right\};\\ E^t_{2,\lambda}=\left\{x\in\mathbb{R}^n\setminus B(0,\epsilon_1):\ \max\left\{\lambda,\dfrac{1}{C(B(x,|x|-\epsilon_t))}\right\}<M_C\nu_t(x)\right\},\end{cases}$$

then $E^t_{1,\lambda}\cup E^t_{2,\lambda}\cup B(0,\epsilon_1)=\{x\in\mathbb{R}^n:\ M_C\nu_t(x)>\lambda\}$.

On the one hand, for such $x\in E^t_{2,\lambda}$ and $\forall\tilde r>0$, one has:

$$\frac{\nu_t(B(x,\tilde r))}{C(B(x,|x|-\epsilon_t))}\leqslant\frac{1}{C(B(x,|x|-\epsilon_t))}<M_C\nu_t(x).$$

Additionally, since for any $r_1,\ r_2$ satisfying $0\leqslant r_1\leqslant r_2$ one has $C(B(x,r_1))\leqslant C(B(x,r_2))$, one gets $C(B(x,r))$ is an increasing function with respect to r. There exists $r<|x|-\epsilon_t$ such that:

$$\frac{\nu_t(B(x,r))}{C(B(x,|x|-\epsilon_t))}\leqslant\frac{\nu_t(B(x,r))}{C(B(x,r))}\leqslant M_C\nu_t(x),$$

and hence by Assumption 1, for any $x_i\in E^t_{2,\lambda}$ there exists $r_i>0$ such that $r_i<|x_i|-\epsilon_t$ and $\lambda\leqslant\nu_t(B(x_i,r_i))/C(B(x,r))$. By the Wiener covering lemma, there exists a disjoint collection of such balls $B_i=B(x_i,r_i)$ and a constant $\alpha>0$ such that $\bigcup_i B_i\subseteq E^t_{2,\lambda}\subseteq\bigcup_i\alpha B_i$. Therefore, we get a constant $\gamma>0$, which only depends on α, such that:

$$C\big(E^t_{2,\lambda}\big)\leqslant\gamma\sum_i C(B_i)<\gamma\sum_i\frac{\nu_t(B_i)}{\lambda}\leqslant\frac{\gamma\epsilon}{\lambda},$$

thanks to $B_i\cap B(0,\epsilon_t)=\emptyset$ and $1-\nu_t(B(0,\epsilon_t))<\epsilon$.

On the other hand, if $x\in E^t_{1,\lambda}$, then:

$$\frac{1-\epsilon}{C(B(x,|x|+\epsilon_t))}\leqslant\frac{\nu_t(B(x,|x|+\epsilon_t))}{C(B(x,|x|+\epsilon_t))}\leqslant M_C\nu_t(x)\leqslant\frac{1}{C(B(x,|x|-\epsilon_t))}.$$

Since

$$\lim_{t\to 0}\left(\frac{1}{C(B(x,|x|+\epsilon_t))}-\frac{1}{C(B(x,|x|-\epsilon_t))}\right)=0\quad\text{and}\quad\lim_{t\to 0}\left(\frac{1}{C(B(x,|x|+\epsilon_t))}-\frac{1}{C(B(x,|x|))}\right)=0,$$

for $\eta>0$ there exists $T>0$ such that:

$$\left|M_C\nu_t(t)-M_C\delta_0\right|<\eta+\frac{\epsilon}{C(B(0,|x|))}<\eta+\frac{\epsilon}{C(B(0,\epsilon_1))}<2\eta\quad\forall t\in(0,T).$$

Note that:

$$M_C\delta_0(x)-2\eta\leqslant M_C\nu_t\leqslant M_C\delta_0(x)+2\eta\quad\forall x\in E^t_{1,\lambda}.$$

Thus:

$$\big\{x\in\mathbb{R}^n:\ M_C\delta_0(x)>\lambda+2\eta\big\}\subseteq E^t_{1,\lambda}\subseteq\big\{x\in\mathbb{R}^n:\ M_C\delta_0(x)>\lambda+2\eta\big\}.$$

This in turn implies:

$$C\big(\big\{x\in\mathbb{R}^n:\ M_C\delta_0(x)>\lambda+2\eta\big\}\big)\leqslant C\big(E^t_{1,\lambda}\big)\leqslant C\big(\big\{x\in\mathbb{R}^n:\ M_C\delta_0(x)>\lambda+2\eta\big\}\big).$$

Now, an application of Lemma 2.1 yields:

$$\frac{1}{\lambda+2\eta}\leqslant C\big(\big\{x\in\mathbb{R}^n:\ M_C\nu_t(x)>\lambda\big\}\cap\big(\mathbb{R}^n\setminus B(0,\epsilon_1)\big)\big)\leqslant\frac{1}{\lambda-2\eta}+\frac{\gamma\epsilon}{\lambda}.$$

Letting $t\to 0$ and using Assumption 1, we get $\lim_{t\to 0}C(\{x\in\mathbb{R}^n:\ M_C\nu_t(x)>\lambda\})=\lambda^{-1}$. □

Lemma 2.3. *If ν is a nonnegative Borel measure on $\mathbb{R}^n$, then $M_C\nu(x)$ is upper semi-continuous.*

Proof. According to the definition of $M_C\nu(x)$, there exists a radius r corresponding to $M_C\nu(x) > \lambda > 0$ such that $\nu(B(x,r))/C(B(x,r)) > \lambda$. For a slightly larger number s with $\lambda + \delta > s > r$, we have $\nu(B(x,r))/C(B(x,s)) > \lambda$. Then applying Assumption 1, one gets that for any z satisfying $|z-x| < \delta$, $M_C\nu(z) \geqslant \nu(B(z,s))/C(B(z,s)) \geqslant \nu(B(x,r))/C(B(x,s)) > \lambda$. whence finding that $\{x \in \mathbb{R}^n\colon\ M_C\nu(x) > \lambda\}$ is open, as desired. □

Lemma 2.4. *If ν is a finite nonnegative Borel measure on $\mathbb{R}^n$, then there exists a constant $\gamma > 0$ such that $\lambda C(\{x \in \mathbb{R}^n\colon\ M_C\nu(x) > \lambda\}) \leqslant \gamma\nu(\mathbb{R}^n)$.*

Proof. Following the argument for [4, p. 39, Theorem 5.6], we set $E_\lambda = \{x \in \mathbb{R}^n\colon\ M_C\nu(x) > \lambda\}$, and then select a ν-measurable set $E \subseteq E_\lambda$ with $\nu(E) < \infty$. Lemma 2.3 proves that E_λ is open. Therefore, for each $x \in E$, there exists an x-related ball B_x such that $\nu(B_x)/C(B_x) > \lambda$. A slight modification of the proof of [4, p. 39, Lemma 5.7] applied to the collection of balls $\{B_x\}_{x\in E}$, and Assumption 2 show that we can find a sub-collection of disjoint balls $\{B_i\}$ and a constant $\gamma > 0$ such that:

$$C(E) \leqslant \gamma \sum_i C(B_i) \leqslant \sum_i \frac{\gamma}{\lambda}\nu(B_i) \leqslant \frac{\gamma}{\lambda}\nu\big(\mathbb{R}^n\big).$$

Note that E is an arbitrary subset of E_λ. Thereby, we can take the supremum over all such E and then get $C(E_\lambda) < (\gamma/\lambda)\nu(\mathbb{R}^n)$. □

3. Proof of theorem

First of all, suppose that ν is a finite nonnegative Borel measure on $\mathbb{R}^n$ with $\nu(\mathbb{R}^n) = 1$. According to the definition of the capacitary maximal function, we have:

$$M_C\nu_t(x) = \sup_{r>0} \frac{\nu_t(B(x,r))}{C(B(x,r))} = \sup_{r>0} \frac{\nu(B(\frac{x}{t},\frac{r}{t}))}{C(tB(\frac{x}{t},\frac{r}{t}))}.$$

From Assumption 2 it follows that:

$$\frac{M_C\nu(\frac{x}{t})}{\psi(t)} \leqslant M_C\nu_t(x) \leqslant \frac{M_C\nu(\frac{x}{t})}{\phi(t)},$$

and so that

$$\left\{x \in \mathbb{R}^n\colon\ M_C\nu\left(\frac{x}{t}\right) > \lambda\psi(t)\right\} \subseteq \big\{x \in \mathbb{R}^n\colon\ M_C\nu_t(x) > \lambda\big\} \subseteq \left\{x \in \mathbb{R}^n\colon\ M_C\nu\left(\frac{x}{t}\right) > \lambda\phi(t)\right\}.$$

The last inclusions give that:

$$\begin{aligned}
\frac{\phi(t)}{\psi(t)}\lambda\psi(t)C\big(\big\{x \in \mathbb{R}^n\colon\ M_C\nu(x) > \lambda\psi(t)\big\}\big) &\leqslant \lambda\phi(t)C\big(\big\{x \in \mathbb{R}^n\colon\ M_C\nu(x) > \lambda\psi(t)\big\}\big)\\
&\leqslant \lambda C\big(\big\{tx \in \mathbb{R}^n\colon\ M_C\nu(x) > \lambda\psi(t)\big\}\big)\\
&= \lambda C\big(\big\{x \in \mathbb{R}^n\colon\ M_C\nu(x/t) > \lambda\psi(t)\big\}\big)\\
&\leqslant \lambda C\big(\big\{x \in \mathbb{R}^n\colon\ M_C\nu_t(x) > \lambda\big\}\big)\\
&\leqslant \lambda C\big(\big\{x \in \mathbb{R}^n\colon\ M_C\nu(x/t) > \lambda\phi(t)\big\}\big)\\
&= \lambda C\big(\big\{tx \in \mathbb{R}^n\colon\ M_C\nu(x) > \lambda\phi(t)\big\}\big)\\
&\leqslant \lambda\psi(t)C\big(\big\{x \in \mathbb{R}^n\colon\ M_C\nu(x) > \lambda\phi(t)\big\}\big)\\
&\leqslant \frac{\psi(t)}{\phi(t)}\lambda\phi(t)C\big(\big\{x \in \mathbb{R}^n\colon\ M_C\nu(x) > \lambda\phi(t)\big\}\big).
\end{aligned}$$

These estimates and Lemma 2.2, plus applying Assumption 2 and letting $t \to 0$, in turns derive:

$$\tau^{-1} \leqslant \liminf_{\lambda\to 0} \lambda C\big(\big\{x \in \mathbb{R}^n\colon\ M_C\nu(x) > \lambda\big\}\big) \leqslant \limsup_{\lambda\to 0} \lambda C\big(\big\{x \in \mathbb{R}^n\colon\ M_C\nu(x) > \lambda\big\}\big) \leqslant \tau.$$

Next, let $h(\lambda) = \lambda C(\{x \in \mathbb{R}^n\colon\ M_C\nu > \lambda\})$. By Lemma 2.4 and the last estimate for both the limit inferior and the limit superior, there exist two constants $A > 0$ and $\lambda_0 > 0$ such that $A \leqslant h(\lambda) \leqslant \gamma\ \forall\lambda \in (0, \lambda_0)$. Moreover, for any given $\varepsilon > 0$, choose a sequence $\{y_i = [\frac{\gamma}{A}(1-\varepsilon)^N]^i\}_1^\infty$, where N is a natural number satisfying $\frac{\gamma}{A}(1-\varepsilon)^N < 1$. Then, there exists an integer $N_0 \geqslant 1$, such that $y_{N_0} < \lambda_0$. Hence, for any $n > m > N_0$, we have:

$$\begin{aligned}
\left|h(y_m)-h(y_n)\right| &\leqslant \left|y_m C\left(\left\{x\in\mathbb{R}^n\colon M_C\nu(x)>y_m\right\}\right)-y_n C\left(\left\{x\in\mathbb{R}^n\colon M_C\nu(x)>y_n\right\}\right)\right| \\
&\leqslant |y_m-y_n| C\left(\left\{x\in\mathbb{R}^n\colon M_C\nu(x)>y_m\right\}\right) \\
&\quad + y_n\left|C\left(\left\{x\in\mathbb{R}^n\colon M_C\nu(x)>y_m\right\}\right)-C\left(\left\{x\in\mathbb{R}^n\colon M_C\nu(x)>y_n\right\}\right)\right| \\
&\leqslant |y_m-y_n|\frac{\gamma}{y_m}+y_n\left|\frac{\gamma}{y_n}-\frac{A}{y_m}\right| \\
&\leqslant \gamma\left(1-\frac{y_n}{y_m}\right)+\left(\gamma-A\frac{y_n}{y_m}\right) \\
&\leqslant \gamma\left(1-\left[\frac{\gamma}{A}(1-\varepsilon)^N\right]^{n-m}\right)+\left(\gamma-A\left[\frac{\gamma}{A}(1-\varepsilon)^N\right]^{n-m}\right) \\
&\leqslant \gamma\left(1-(1-\varepsilon)^{N(n-m)}\right)+\left(\gamma-\gamma(1-\varepsilon)^{N(n-m)}\right) \\
&\leqslant 2\gamma N(n-m)\varepsilon.
\end{aligned}$$

Consequently, $\{h(y_i)\}$ is a Cauchy sequence, $D=\lim_{i\to\infty}h(y_i)$ exists. Note that for any small λ, there exists a large i such that $y_{i+1}\leqslant\lambda\leqslant y_i$. Thereby, from the triangle inequality, it follows that if i is large enough, then:

$$\begin{aligned}
\left|h(\lambda)-D\right| &\leqslant \left|h(\lambda)-h(y_i)\right|+\left|h(y_i)-D\right| \\
&\leqslant |y_i-\lambda|\frac{\gamma}{y_i}+\lambda\left|\frac{\gamma}{\lambda}-\frac{A}{y_i}\right|+\left|h(y_i)-D\right| \\
&\leqslant \gamma\left(1-\frac{\lambda}{y_i}\right)+\left(\gamma-A\frac{\lambda}{y_i}\right)+\left|h(y_i)-D\right| \\
&\leqslant \gamma\left(1-\frac{y_{i+1}}{y_i}\right)+\left(\gamma-A\frac{y_{i+1}}{y_i}\right)+\left|h(y_i)-D\right| \\
&\leqslant (2\gamma N+1)\varepsilon.
\end{aligned}$$

This in turn implies that $\lim_{\lambda\to0}\lambda C(\{x\in\mathbb{R}^n\colon M_C\nu(x)>\lambda\})$ exists, and consequently, $\tau^{-1}\leqslant\lim_{\lambda\to0}\lambda C(\{x\in\mathbb{R}^n\colon M_C\nu(x)>\lambda\})\leqslant\tau$ holds.

Finally, upon employing the given $L^1(\mathbb{R}^n)$ function f with $\|f\|_1>0$ to produce a finite nonnegative measure ν with $\nu(\mathbb{R}^n)=1$ via

$$\nu(E)=\frac{1}{\|f\|_1}\int_E\left|f(y)\right|\mathrm{d}y\quad\forall E\subseteq\mathbb{R}^n,$$

we obtain:

$$\lim_{\lambda\to0}\lambda C\left(\left\{x\in\mathbb{R}^n\colon M_C f(x)>\lambda\|f\|_1\right\}\right)\approx1,$$

thereby getting:

$$\lim_{\lambda\to0}\lambda\|f\|_1 C\left(\left\{x\in\mathbb{R}^n\colon M_C f(x)>\lambda\|f\|_1\right\}\right)\approx\|f\|_1.$$

By setting $\tilde{\lambda}=\lambda\|f\|_1$ in the last estimate, we reach the desired result.

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