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New Poincaré-type inequalities

Quelques inégalités de type Poincaré pour les champs de matrices quadratiques

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ABSTRACT

We present some Poincaré-type inequalities for quadratic matrix fields with applications e.g. in gradient plasticity or fluid dynamics. In particular, applications to the pseudostress–velocity formulation of the stationary Stokes problem and to infinitesimal gradient plasticity are discussed.

RÉSUMÉ

On présente quelques inégalités de type Poincaré pour les champs de matrices quadratiques, avec des applications, par exemple, en plasticité avec gradients ou en dynamique des fluides. En particulier, on discute des applications pour la formulation en vitesse de pseudo-tension du problème stationnaire de Stokes et pour la plasticité infinitésimale avec gradients.

1. Results: dev–Div–sym–Curl estimates

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Moreover, let $\Gamma \neq \emptyset$ be a relatively open subset of $\partial\Omega$. The usual L^2-Sobolev spaces for the gradient, rotation and divergence with homogeneous scalar, tangential resp. normal trace on Γ will be denoted by $H^1_\Gamma(\Omega)$, $H_\Gamma(\operatorname{curl},\Omega)$, $H_\Gamma(\operatorname{div},\Omega)$, respectively. A matrix-valued function $T:\Omega\to\mathbb{R}^{3\times3}$ belongs to the Sobolev space $H_\Gamma(\operatorname{Curl},\Omega)$ resp. $H_\Gamma(\operatorname{Div},\Omega)$, if its rows are elements of $H_\Gamma(\operatorname{curl},\Omega)$ resp. $H_\Gamma(\operatorname{div},\Omega)$, and the differential operators Curl and Div act row-wise as curl and div. Moreover, we will frequently use the standard matrix operations $\operatorname{sym} T = \frac{1}{2}(T+T^\top)$, $\operatorname{skew} T = \frac{1}{2}(T-T^\top)$ and $\operatorname{dev} T = T - \operatorname{tr} T/3\cdot\operatorname{id}$, where id denotes the identity matrix. The standard L^2-norms and scalar products for scalar, vector- or matrix-valued functions are denoted by $|\cdot|$ and $\langle\cdot,\cdot\rangle$, respectively. The following three inequalities hold:

Theorem 1.1. *There exists a constant $c>0$ such that for all $T\in H_\Gamma(\operatorname{Curl},\Omega)$ the estimate*

$$c|T| \leqslant |\operatorname{dev}\operatorname{sym} T| + |\operatorname{Curl} T|$$

holds true. Hence, on $H_\Gamma(\operatorname{Curl},\Omega)$ the right-hand side of this inequality defines a norm equivalent to the $H(\operatorname{Curl},\Omega)$-norm, i.e.,

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$$|T| + |\operatorname{Curl} T| \cong |\operatorname{dev}\operatorname{sym} T| + |\operatorname{Curl} T|.$$

See also our papers [13–17].

Proof. The proof follows in close lines the paper [15] and relies on three essential tools, namely the Maxwell estimate (a Poincaré-type estimate for curl and div), the Helmholtz decomposition and Korn's first inequality. The only change is to prove a stronger version of Korn's first inequality, namely:

$$\forall v \in H^1_\Gamma(\Omega), \qquad c|\nabla v| \leqslant |\operatorname{dev}\operatorname{sym}\nabla v|, \tag{1}$$

which finishes the proof. Inequality (1) is proved by controlling the kernel of $\operatorname{dev}\operatorname{sym}\nabla$ on $H^1_\Gamma(\Omega)$ and using a stronger version of Korn's second inequality, established in [7] and with a different proof in [11]. For details, see [2]. For a proof of (1) in $W^{1,q}(\Omega)$, $\Omega \subset \mathbb{R}^2$, $1 < q < \infty$ with full boundary condition, see also [9]. $\square$

Theorem 1.2. *There exists a constant $c > 0$ such that for all $T \in H_\Gamma(\operatorname{Div}, \Omega)$ the estimate*

$$c\,|T| \leqslant |\operatorname{dev} T| + |\operatorname{Div} T|$$

holds true. Hence, on $H_\Gamma(\operatorname{Div}, \Omega)$ the right-hand side of this inequality defines a norm equivalent to the $H(\operatorname{Div}, \Omega)$-norm, i.e.,

$$|T| + |\operatorname{Div} T| \cong |\operatorname{dev} T| + |\operatorname{Div} T|.$$

Proof. Let $\tilde{\Gamma} := \partial\Omega \setminus \overline{\Gamma}$ be the complement of Γ. Following [21], we first prove:

$$\exists c > 0, \quad \forall f \in L^2(\Omega), \quad \exists v \in H^1_{\tilde{\Gamma}}(\Omega), \qquad \operatorname{div} v = f, \quad |v| + |\nabla v| \leqslant c|f|.$$

Then, we utilize the idea of [1, Lemma 3.1] and obtain with some $v \in H^1_{\tilde{\Gamma}}(\Omega)$ solving $\operatorname{div} v = \operatorname{tr} T$:

$$\begin{aligned} |\operatorname{tr} T|^2 &= \langle \operatorname{tr} T, \operatorname{div} v\rangle = \langle \operatorname{tr} T, \operatorname{tr}\nabla v\rangle = \langle \operatorname{tr} T \cdot \mathrm{id}, \nabla v\rangle = 3\langle T, \nabla v\rangle - 3\langle \operatorname{dev} T, \nabla v\rangle \\ &= -3\langle \operatorname{Div} T, v\rangle - 3\langle \operatorname{dev} T, \nabla v\rangle \leqslant c\big(|\operatorname{dev} T| + |\operatorname{Div} T|\big)|\operatorname{tr} T|, \end{aligned} \tag{2}$$

which completes the proof, since it is sufficient to estimate $\operatorname{tr} T$. $\square$

Corollary 1.3. *There exists a constant $c > 0$ such that for all $T \in H_\Gamma(\operatorname{Curl}, \Omega)$ the estimate*

$$c\,|T| \leqslant |\operatorname{dev}\operatorname{sym} T| + |\operatorname{dev}\operatorname{Curl} T|$$

holds true. Again, on $H_\Gamma(\operatorname{Curl}, \Omega)$ the right-hand side of this inequality defines a norm equivalent to the $H(\operatorname{Curl}, \Omega)$-norm, i.e.,

$$|T| + |\operatorname{Curl} T| \cong |\operatorname{dev}\operatorname{sym} T| + |\operatorname{dev}\operatorname{Curl} T|.$$

Proof. Since $S := \operatorname{Curl} T \in H_\Gamma(\operatorname{Div}, \Omega)$ with $\operatorname{Div} S = 0$ the assertion follows by a combination of Theorem 1.1 and Theorem 1.2. $\square$

2. Application to the pseudostress–velocity formulation of the stationary Stokes problem

Let us study the following first-order system formulation of the stationary Stokes equations: For some given vector field f in $L^2(\Omega)$, find a scalar function $p \in L^2(\Omega)$, a vector-valued function $u \in H^1_\Gamma(\Omega)$ and a matrix-valued function $\sigma \in H_{\tilde{\Gamma}}(\operatorname{Div}, \Omega)$, such that the system:

$$\sigma - \mu\operatorname{sym}\nabla u + p \cdot \mathrm{id} = 0, \qquad \operatorname{Div}\sigma = f, \qquad \operatorname{div} u = 0$$

holds in Ω. Such a formulation is of interest from a numerical viewpoint if accurate stress approximations in appropriate finite-element spaces are desired. This system is equivalent to:

$$\operatorname{dev}\sigma - \mu\operatorname{sym}\nabla u = 0, \qquad \operatorname{Div}\sigma = f,$$

where the pressure p has been eliminated and can be computed afterwards as $p = -\operatorname{tr}\sigma/3$. For this first-order system, a least-squares formulation based on minimizing the quadratic functional:

$$(u, \sigma) \mapsto |\operatorname{dev}\sigma - \mu\operatorname{sym}\nabla u|^2 + |\operatorname{Div}\sigma - f|^2$$

with respect to $(u, \sigma) \in H^1_\Gamma(\Omega) \times H_{\tilde{\Gamma}}(\operatorname{Div}, \Omega)$ was studied in [4, Section 3.2]. The well-posedness of this least squares problem is shown based on a coercivity result of the form:

$$|\operatorname{dev}\sigma-\mu\operatorname{sym}\nabla u|^2+|\operatorname{Div}\sigma|^2\geqslant c\big(|\operatorname{Div}\sigma|^2+|\sigma|^2+|\nabla u|^2+|u|^2\big)=c\big(|u|^2_{H^1(\Omega)}+|\sigma|^2_{H(\operatorname{Div},\Omega)}\big) \tag{3}$$

to hold with a constant $c>0$ for all $(u,\sigma)\in H^1_\Gamma(\Omega)\times H_{\tilde\Gamma}(\operatorname{Div},\Omega)$. In order to obtain (3) and since the $H^1(\Omega)$-norm of u is controlled by $|\operatorname{sym}\nabla u|$ using Korn's first and Poincaré's inequalities, σ, more precisely $\operatorname{tr}\sigma$, needs to be controlled by the first-order system, i.e., the result of Theorem 1.2 is required. The inequality (3) is then proved in a way similar to the ellipticity proof in [5, Theorem 3.1] using:

$$\begin{aligned}\langle\operatorname{dev}\sigma,\operatorname{sym}\nabla u\rangle&=\langle\operatorname{sym}\operatorname{dev}\sigma,\nabla u\rangle=\Big\langle\operatorname{sym}\sigma-\frac13\operatorname{tr}\sigma\cdot\mathrm{id},\nabla u\Big\rangle\\&=\langle\sigma,\nabla u\rangle-\langle\operatorname{skew}\sigma,\nabla u\rangle-\frac13\langle\operatorname{tr}\sigma,\operatorname{div}u\rangle\\&=-\langle\operatorname{Div}\sigma,u\rangle-\langle\operatorname{skew}\sigma,\nabla u\rangle-\frac13\langle\operatorname{tr}\sigma,\operatorname{div}u\rangle\end{aligned}$$

and

$$\begin{aligned}|\operatorname{skew}\sigma|&=\big|\operatorname{skew}(\operatorname{dev}\sigma-\mu\operatorname{sym}\nabla u)\big|\leqslant|\operatorname{dev}\sigma-\mu\operatorname{sym}\nabla u|,\\\mu|\operatorname{div}u|&=\big|\operatorname{tr}(\operatorname{dev}\sigma-\mu\operatorname{sym}\nabla u)\big|\leqslant\sqrt3|\operatorname{dev}\sigma-\mu\operatorname{sym}\nabla u|.\end{aligned}$$

In [5], a result similar to Theorem 1.2 has been obtained indirectly by examination of the incompressible limit of a first-order system linear elasticity formulation.

A widely used result in the literature on mixed methods is for the inequality of Theorem 1.2 to hold for all $\sigma\in H(\operatorname{Div}\Omega)$ which satisfy $\langle\operatorname{tr}\sigma,1\rangle=0$. This result dates back to [1], see also [3, Section IV.3] and is useful in the mixed framework where the boundary conditions on the normal stress are only treated weakly. Such mixed approaches have been analyzed recently for the stationary Stokes problem in [6] and [10].

3. Application to infinitesimal gradient plasticity

Phenomenological plasticity models are intended to describe the irreversible deformation behavior of metals. There exists a wide variety of models. Here we focus on rate-dependent or rate-independent models with kinematic hardening. The system of equations consists of a balance of a linear momentum coupled with a local nonlinear evolution equation in each space point for the plastic variable.

In many new applications, the size of the considered specimen is so small that size effects need to be taken into account. Instead of a local evolution problem, we have to consider a nonlinear evolution problem where the right-hand side contains certain combinations of second partial derivatives of the plastic distortion.

In more detail, let our domain Ω denote the set of material points of the solid body. By T_e we denote a positive number (time of existence). Unknown in our small strain formulation are the displacement $u:\Omega\times[0,T_\mathrm{e})\to\mathbb{R}^3$ of the material point x at time t and the not necessarily symmetric infinitesimal plastic distortion $P:\Omega\times[0,T_e)\to\mathfrak{sl}(3)$. The model equations of the problem are:

$$\begin{aligned}&-\operatorname{Div}\sigma=b,\qquad \Sigma^{\mathrm{lin}}=\Sigma^{\mathrm{lin}}_{\mathrm{e}}+\Sigma^{\mathrm{lin}}_{\mathrm{sh}}+\Sigma^{\mathrm{lin}}_{\mathrm{curl}},\quad \Sigma^{\mathrm{lin}}_{\mathrm{e}}=\sigma,\\&\sigma=2\mu\big(\operatorname{sym}(\nabla u-P)\big)+\lambda\operatorname{tr}\nabla u\cdot\mathrm{id},\qquad \Sigma^{\mathrm{lin}}_{\mathrm{sh}}=-\operatorname{dev}\operatorname{sym}P,\\&\partial_tP\in g\big(x,\Sigma^{\mathrm{lin}}\big),\qquad \Sigma^{\mathrm{lin}}_{\mathrm{curl}}=-\operatorname{Curl}\operatorname{Curl}P,\end{aligned}$$

which must be satisfied in $\Omega\times[0,T_\mathrm{e})$. Here, Σ^{lin} is the infinitesimal Eshelby stress tensor driving the evolution of the plastic distortion P. The initial and boundary conditions are $P(x,0)=P_0(x)\in\mathfrak{sl}(3)$ for all $x\in\Omega$ and $\nu\times P=0$, $u=0$ on $\partial\Omega\times[0,T_\mathrm{e})$, respectively, where ν denotes the outer unit normal on the boundary $\partial\Omega$. For the model we require that the nonlinear constitutive mapping g is monotone. Given are the volume force b and the initial datum P_0. It is easy to see that the corresponding free energy of the system is:

$$\mathcal{E}(u,P)=\mu\big|\operatorname{sym}(\nabla u-P)\big|^2+\frac{\lambda}{2}|\operatorname{tr}\nabla u|^2+\frac12|\operatorname{dev}\operatorname{sym}P|^2+\frac12|\operatorname{Curl}P|^2.$$

The appearance of $|\operatorname{Curl}P|^2$ instead of the full gradient $|\nabla P|^2$ is dictated by dislocation mechanics, the appearance of $|\operatorname{dev}\operatorname{sym}P|^2$ instead of $|P|^2$ is dictated by the invariance of the model under superposition of infinitesimal rotations. Here, coercivity is obtained by using Theorem 1.1. Model equations similar to the above problem have been considered, e.g., in [19,20,8,18,12].

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