C. R. Acad. Sci. Paris, Ser. I 352 (2014) 105–108

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Complex analysis

Yet another proof for a theorem of Landau on Koebe domains

Une preuve supplémentaire d'un théorème de Landau sur les domaines de Koebe

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ARTICLE INFO

Article history:
Received 2 July 2013
Accepted after revision 9 December 2013
Available online 10 January 2014

Presented by the Editorial Board

Dedicated to Ken-ichi Sakan on the occasion of his sixty-fifth birthday and formal retirement from Osaka City University

ABSTRACT

The Koebe domain of a family of functions, holomorphic on the unit disk, is the largest domain that is contained in the image of the unit disk for every function of the family. In this note, we furnish a geometric proof of a classical theorem due to Landau on the Koebe domains for certain families of holomorphic functions. The method of proof involves our recently obtained results concerning estimates for hyperbolic metrics on subdomains.

RÉSUMÉ

Le domaine de Koebe d'une famille de fonctions holomorphes sur le disque unité est le plus grand domaine qui est contenu dans l'image du disque par chaque fonction de la famille. Dans cette note, nous présentons une preuve géométrique d'un théorème classique de Landau, relatif aux domaines de Koebe de certaines familles de fonctions holomorphes. La méthode de preuve met en jeu notre résultats récents concernant les estimations pour les métriques hyperboliques sur des sous-domaines.

Suppose that $f(z)$ is a holomorphic function on the unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$, represented by a power series:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots.$$

The holomorphic map given by such $f(z)$ maps $\mathbb{D}$ onto some subdomain of $\mathbb{C}$ when $f(z)$ is a non-constant function.

The Koebe domain for a family $\mathfrak{F}$ of these functions $f(z)$ is denoted by $K(\mathfrak{F})$ and, by definition, this is the largest domain contained in $f(\mathbb{D})$ for every function $f(z)$ in $\mathfrak{F}$.

Goodman [1] has studied the Koebe domains $K(\mathfrak{F})$ for various families $\mathfrak{F}$, containing survey of known Koebe domains. In the present paper, we are concerned with the Koebe domains for a certain class of the families Goodman treated in [1] first among the said various families, which will be explained right below.

Let $\mathfrak{B}(A)$ be the family of bounded holomorphic functions on $\mathbb{D}$

$$f(z) = Az + \sum_{n=2}^{\infty} a_n z^n, \quad A > 0,$$

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1631-073X/$ – see front matter
http://dx.doi.org/10.1016/j.crma.2013.12.009

satisfying $|f(z)| < 1$. We note that, since $f(0) = 0$ is assumed, the Schwarz lemma immediately implies that $A \leqslant 1$ with equality if and only if $f(z) = z$. These assumptions are not essential and could be looked upon as the result of a normalization.

Theorem. *(See also Goodman [1].) Let $\delta = r(A)$ be the solution of the equation*:

$$A = \frac{-2\delta \log \delta}{1 - \delta^2}, \quad 0 < A < 1, \tag{1}$$

which satisfies $0 < r(A) < 1$. Then the Koebe domain $K(\mathfrak{B}(A))$ is the disk $\{|z| < r(A)\}$. Moreover, a point z_1 on the boundary $\partial K(\mathfrak{B}(A))$ is omitted only by an extremal function $f(z)$ in $\mathfrak{B}(A)$ that gives an unramified holomorphic universal covering map $\mathbb{D} \to \dot{\mathbb{D}}(z_1)$, where $\dot{\mathbb{D}}(z_1)$ is a once punctured unit disk $\mathbb{D} \setminus \{z_1\}$.

Under our normalization assumption, a prescribed boundary point $z_1 \in \partial K(\mathfrak{B}(A))$, which specifies the domain $\dot{\mathbb{D}}(z_1)$, uniquely yields the extremal function $f(z) \in \mathfrak{B}(A)$.

In fact, Goodman [1] in 1979 states that Landau [4] in 1929 already determined $K(\mathfrak{B}(A))$ and that his proof of the same result is somewhat different and slightly shorter, compared with Landau's proof. We will furnish yet another geometric proof which involves our recently obtained result concerning lower estimates for hyperbolic (or Poincaré) metrics on subdomains.

Let $d\sigma_D(z) = \rho_D(z)\,|dz|$ denote the hyperbolic metric on a hyperbolic planar domain D. In what follows we let:

$$\rho_{\mathbb{D}}(z) = \frac{2}{1 - |z|^2}, \quad z \in \mathbb{D}.$$

From this metric, we get the distance function $d_{\mathbb{D}}(\cdot, \cdot)$ in a usual way by integrating along curves between two end points and taking the infimum. It is to be recalled that:

$$d_{\mathbb{D}}(z, 0) = \log \frac{1 + |z|}{1 - |z|}, \quad z \in \mathbb{D}. \tag{2}$$

A fundamental property of the hyperbolic metric, known from the Schwarz–Pick lemma, is the contracting property for holomorphic maps stating that if a holomorphic function $f(z)$ gives a holomorphic map:

$$f : D_1 \to D_2$$

between two hyperbolic planar domains D_1 and D_2, then

$$\rho_{D_2}\big(f(z)\big)\big|f'(z)\big| \leqslant \rho_{D_1}(z), \quad z \in D_1 \tag{3}$$

with strict inequality unless $f(z)$ gives an unramified holomorphic covering map.

In order to relate the problem to the hyperbolic metric, we do need the following proposition, in which the curious function $(-2\delta \log \delta)/(1 - \delta^2)$ of the Theorem appears at the left-hand side of our lower estimate.

Proposition. *Let D be a nonempty subdomain of the unit disk $\mathbb{D}$. Then, for $z \in D$,*

$$\frac{1 - \delta_D(z)^2}{-2\delta_D(z) \log \delta_D(z)} \leqslant \frac{\rho_D(z)}{\rho_{\mathbb{D}}(z)} \tag{4}$$

where $\delta_D(z)$ is a real-valued function on D satisfying $0 < \delta_D(z) \leqslant 1$ that is determined by:

$$d_{\mathbb{D}}(z, \partial D) = \log \frac{1 + \delta_D(z)}{1 - \delta_D(z)}. \tag{5}$$

If equality holds in (4) at some point in a proper subdomain D of $\mathbb{D}$, then D is a once punctured unit disk $\dot{\mathbb{D}}(z_1) = \mathbb{D} \setminus \{z_1\}$ for $z_1 \in \mathbb{D}$.

Remark. By introducing a minor convention that the left-hand side quantity of the inequality (4) reduces to the constant 1 in the limiting case corresponding to $\delta_D(z) = 1$, it is possible to make no exception for the nonproper subdomain.

The above proposition is proved by an application of the "comparison principle" for hyperbolic metrics, which can be derived from the Schwarz–Pick lemma, to an appropriate once punctured unit disk that can serve as a reference subdomain.

Indeed, although there is no explicit formula in general for the density $\rho_D(z)$ even on a planar domain D, in a simple case where D is $\dot{\mathbb{D}} = \dot{\mathbb{D}}(0)$, it is well known and easy to verify that:

$$\rho_{\dot{\mathbb{D}}}(z) = \frac{1}{|z| \log \frac{1}{|z|}}, \quad z \in \dot{\mathbb{D}} \tag{6}$$

since, when we work with the upper half plane (instead of the unit disk), the exponential function gives a related universal covering map. For fuller details, we refer the reader to Ito [3], for instance.

We note that $\delta_D(z)$ determined by (5) is the so-called pseudo-hyperbolic distance from z to the boundary ∂D. In an important sense our starting point is the relation

$$\delta_D(0) = \text{Euclidean distance from 0 to the boundary } \partial D \tag{7}$$

resulting from the familiar formula (2): namely, when D contains the origin, $\delta_D(0)$ is just the radius of the largest open disk around 0 that is contained in D.

Proof of Theorem. Let A be a parameter with $0 < A < 1$. For an arbitrarily chosen function $f(z)$ in $\mathfrak{B}(A)$, we shall, in the sequel, denote simply by D the image under $f(z)$ of the unit disk $\mathbb{D}$. Recall that $f(z)$ is normalized so that $f(0) = 0$, for example.

The contracting property (3) yields:

$$\frac{\rho_D(0)}{\rho_{\mathbb{D}}(0)} \leqslant \frac{1}{|f'(0)|} = \frac{1}{A}$$

with equality if and only if $f(z)$ gives an unramified holomorphic universal covering map $\mathbb{D} \to D$.

By combining this inequality with the estimate (4) at $z = 0$, we see that:

$$A \leqslant \frac{\rho_{\mathbb{D}}(0)}{\rho_D(0)} \leqslant \frac{-2\delta_D(0)\log\delta_D(0)}{1-\delta_D(0)^2}.$$

In particular, we have:

$$A \leqslant \frac{-2\delta_D(0)\log\delta_D(0)}{1-\delta_D(0)^2}. \tag{8}$$

Thus the Koebe domain $K(\mathfrak{B}(A))$ must contain the disk $\{|z| < r(A)\}$ from the relation (7) since $\delta_D(0)$ has to be equal to or greater than the radius $r(A)$ defined by Eq. (1), because the function $(-2\delta\log\delta)/(1-\delta^2)$ is a strictly increasing function of δ (see also the remark following the proof).

For the purpose of completing the proof, it would be sufficient to investigate the situation where equality holds in the inequality (8).

If equality holds in (8), then $\delta_D(0) = r(A)$ and equality must hold in both of the preceding inequalities. Equality in the latter implies that D is a once punctured unit disk $\dot{\mathbb{D}}(z_1)$ obviously with $|z_1| = r(A)$ by the Proposition, and equality in the former further requires, as remarked before, that $f(z)$ give an unramified holomorphic universal covering map $\mathbb{D} \to \dot{\mathbb{D}}(z_1)$. Therefore, the Koebe domain $K(\mathfrak{B}(A))$ of course cannot be larger than the disk $\{|z| < r(A)\}$. This completes the proof. □

Remark. Goodman [1] mentions Landau [4, p. 620] for an identity:

$$-\frac{2\delta\log\delta}{1-\delta^2} = 1 - \sum_{n=1}^{\infty}\frac{2u^{2n}}{4n^2-1} \tag{9}$$

where $\delta = (1-u)/(1+u)$. Definitely, the right-hand side of (9) is strictly decreasing in u.

We have obtained a "one-line" proof of a theorem of Landau in one fell swoop, which readily detects the extremal functions.

Finally, we remark that for our purposes here the action of a Fuchsian group (possibly with torsion) on $\mathbb{D}$ as in Ito [3] does not need to be considered.

In fact, Goodman [1] has gone further to study the generalized family $\mathfrak{B}_k(A)$ of bounded holomorphic functions on $\mathbb{D}$:

$$f(z) = Az^k + \sum_{n=k+1}^{\infty} a_n z^n, \quad A > 0,$$

satisfying $|f(z)| < 1$. Here k is a positive integer, not necessarily equal to 1.

For any function $f(z)$ in $\mathfrak{B}_k(A)$, there exists a function $\widetilde{f}(z) = A^{1/k}z + \cdots$ in a neighborhood of the origin such that:

$$f(z) = (p_k \circ \widetilde{f})(z), \quad z \in \mathbb{D},$$

where $p_k(z)$ denotes a polynomial function:

$$p_k(z) = z^k.$$

The function p_k is associated with a Fuchsian group Γ generated by an elliptic element of order k that has a fixed point at the origin, and gives a ramified holomorphic universal covering map from $\mathbb{D}$ onto the hyperbolic orbifold $M = \mathbb{D}/\Gamma$.

Therefore, our form of the solution in Ito [2,3] concerning estimates on hyperbolic orbifolds (rather than ordinary Riemann surfaces) may be useful to study the family $\mathfrak{B}_k(A)$.

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