C. R. Acad. Sci. Paris, Ser. I 352 (2014) 95–97

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Mathematical analysis

An extremal problem for polynomials

Un problème extrémal pour les polynômes

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A R T I C L E   I N F O

Article history:
Received 23 March 2013
Accepted 12 December 2013
Available online 31 December 2013

Presented by Jean-Pierre Kahane

A B S T R A C T

We give a solution to an extremal problem for polynomials, which asks for complex numbers $\alpha_0, \dots, \alpha_n$ of unit magnitude that minimise the largest supremum norm on the unit circle for all polynomials of degree n whose k-th coefficient is either α_k or $-\alpha_k$.

R É S U M É

Nous donnons dans ce papier une solution à un problème extrémal sur les polynômes, qui est de trouver des nombres complexes $\alpha_0, \dots, \alpha_n$ de module égal à 1 qui minimisent, sur le cercle unité, la plus grande borne supérieure de la norme pour tous les polynômes de degré n qui ont pour k^{e} coefficient α_k ou $-\alpha_k$.

1. Results

Extremal problems for polynomials are typically of the following form. Let $f(z)$ be a polynomial with coefficients restricted to be in a subset of the complex numbers (often, this set is $\{-1, 1\}$). How well can $|f(z)|$ approximate a constant function when z ranges over the unit circle? This meta problem has many variations, most of which are open (see, for example, Littlewood [8], Borwein [1], and Erdélyi [2] for surveys on selected problems). To quantify the gap between $|f(z)|$ and a constant function, different norms on the unit circle have been considered. The supremum norm:

$$\|f\| = \max_{|z|=1} \left| f(z) \right|$$

has received particular attention. For example, Erdös [3, Problem 22], [4], and Littlewood [7] were interested in the minimum of $\|f_n\|$, where f_n is a polynomial of degree $n-1$ with coefficients of absolute value 1. In particular, Erdös [4] conjectured that there is some $c > 0$ such that $\|f_n\|/\sqrt{n} \geqslant 1 + c$ for all polynomials f_n of degree $n-1$ (for every $n \geqslant 1$), whose coefficients have absolute value 1. Kahane [5] proved that there is no such c, but the modified conjecture where f_n is restricted to have coefficients 1 or -1 only remains open.

In this paper, we study an extremal problem in which the goal is to minimise the largest supremum norm for polynomials whose k-th coefficient is either α_k or $-\alpha_k$, where the minimisation is over the complex numbers $\alpha_0, \alpha_1, \dots$ satisfying $|\alpha_0| = |\alpha_1| = \cdots = 1$. Specifically, defining:

$$B_n(\phi_0, \dots, \phi_{n-1}) = \max_{\epsilon_0, \dots, \epsilon_{n-1} \in \{-1,1\}} \left\| \sum_{k=0}^{n-1} \epsilon_k e^{2\pi i \phi_k} z^k \right\|$$

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1631-073X/$ – see front matter
http://dx.doi.org/10.1016/j.crma.2013.12.011

for $\phi_0, \ldots, \phi_{n-1} \in [0, 1]$, we are interested in the minimum:

$$b(n) = \min_{\phi_0,\ldots,\phi_{n-1}\in[0,1]} B_n(\phi_0, \ldots, \phi_{n-1}).$$

This minimisation problem is also related to an optimisation problem in communications engineering and, in this context, variations of the problem have been studied by Tarokh and Jafarkhani [10], Litsyn and Wunder [6], and Schmidt [9], among others.

It is easy to see that $b(n) \leqslant n$ and that $b(1) = 1$ and $b(2) = 2$, but the value of $b(n)$ is unknown for all $n \geqslant 3$. One might be tempted to conjecture that $b(n)$ is monotonically increasing and $b(n)/n$ is monotonically decreasing, but this also remains unknown.

Our main result is the following.

Theorem 1. *We have*:

$$\lim_{n\to\infty} \frac{b(n)}{n} = \frac{2}{\pi}.$$

We prove Theorem 1 in Propositions 2 and 3 below. Proposition 2 establishes that $b(n)/n > 2/\pi$ for all $n \geqslant 1$ and Proposition 3 gives explicit constructions of $\phi_0, \ldots, \phi_{n-1}$ for which $B_n(\phi_0, \ldots, \phi_{n-1})/n$ approaches $2/\pi$ arbitrarily closely.

Proposition 2. *For each $n \geqslant 1$, we have*:

$$\frac{b(n)}{n} > \frac{2}{\pi}.$$

Proof. We have:

$$\begin{aligned}
B_n(\phi_0, \ldots, \phi_{n-1}) &\geqslant \max_{\epsilon_0,\ldots,\epsilon_{n-1}\in\{-1,1\}} \left|\sum_{k=0}^{n-1} \epsilon_k e^{2\pi i\phi_k}\right| \\
&= \max_{\epsilon_0,\ldots,\epsilon_{n-1}\in\{-1,1\}} \max_{\psi\in[0,1]} \operatorname{Re}\left(\sum_{k=0}^{n-1} \epsilon_k e^{2\pi i\phi_k} e^{2\pi i\psi}\right) \\
&= \max_{\psi\in[0,1]} \sum_{k=0}^{n-1} \bigl|\cos\bigl(2\pi(\phi_k + \psi)\bigr)\bigr|.
\end{aligned}$$

Since the sum cannot be constant for all ψ, we can further bound this expression as follows:

$$\begin{aligned}
B_n(\phi_0, \ldots, \phi_{n-1}) &> \int_0^1 \sum_{k=0}^{n-1} \bigl|\cos\bigl(2\pi(\phi_k + \psi)\bigr)\bigr| \,d\psi \\
&= \sum_{k=0}^{n-1} \int_0^1 \bigl|\cos(2\pi\psi)\bigr| \,d\psi = \frac{2n}{\pi},
\end{aligned}$$

as required. □

Proposition 3. *Let α be an irrational number and write $\phi_k = \alpha k^2$. Then*

$$\lim_{n\to\infty} \frac{B_n(\phi_0, \ldots, \phi_{n-1})}{n} = \frac{2}{\pi}.$$

Proof. We have:

$$\begin{aligned}
B_n(\phi_0, \ldots, \phi_{n-1}) &= \max_{\epsilon_0,\ldots,\epsilon_{n-1}\in\{-1,1\}} \max_{\theta\in[0,1]} \left|\sum_{k=0}^{n-1} \epsilon_k e^{2\pi i\alpha k^2} e^{2\pi i\theta k}\right| \\
&= \max_{\epsilon_0,\ldots,\epsilon_{n-1}\in\{-1,1\}} \max_{\theta,\psi\in[0,1]} \operatorname{Re}\left(\sum_{k=0}^{n-1} \epsilon_k e^{2\pi i(\alpha k^2 + \theta k + \psi)}\right) \\
&= \max_{\theta,\psi\in[0,1]} \sum_{k=0}^{n-1} \bigl|\cos\bigl(2\pi(\alpha k^2 + \theta k + \psi)\bigr)\bigr|. \qquad (1)
\end{aligned}$$

Since α is irrational, we obtain the following consequence of a celebrated result on equi-distributed sequences modulo 1 due to Weyl [11, Satz 9]:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\mathrm{e}^{2\pi\mathrm{i}m(\alpha k^2+\theta k+\psi)}=0\quad\text{for every integer } m\neq 0.$$

Moreover, the convergence is uniform for all $\theta,\psi\in\mathbb{R}$. It then follows easily [11, pp. 314–315] that:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\left|\cos\bigl(2\pi\left(\alpha k^2+\theta k+\psi\right)\bigr)\right|=\int_0^1|\cos 2\pi x|\,\mathrm{d}x$$

uniformly for all $\theta,\psi\in\mathbb{R}$. Hence, from (1),

$$\begin{aligned}\lim_{n\to\infty}\frac{B_n(\phi_0,\dots,\phi_{n-1})}{n}&=\lim_{n\to\infty}\max_{\theta,\psi\in[0,1]}\frac{1}{n}\sum_{k=0}^{n-1}\left|\cos\bigl(2\pi\left(\alpha k^2+\theta k+\psi\right)\bigr)\right|\\&=\max_{\theta,\psi\in[0,1]}\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\left|\cos\bigl(2\pi\left(\alpha k^2+\theta k+\psi\right)\bigr)\right|\\&=\max_{\theta,\psi\in[0,1]}\int_0^1|\cos 2\pi x|\,\mathrm{d}x=\frac{2}{\pi},\end{aligned}$$

as required. □

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