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On the Hamiltonian structure of the planar steady water-wave problem with vorticity

Sur la structure hamiltonienne du problème des ondes de surface planes stationnaires avec vorticité

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ABSTRACT

We consider the stream-function formulation of the hydrodynamic problem for steady rotational water waves both with and without surface tension. A natural Lagrangian formulation is presented from which (different) Hamiltonian formulations for the two cases are derived by duality theory in the spirit of the Legendre–Fenchel transform. The treatment is systematic and clarifies a recent *ad hoc* approach by Kozlov and Kuznetsov [7].

RÉSUMÉ

Nous considérons la formulation de la fonction du courant dans le problème hydrodynamique décrivant les ondes de surface rotationnelles stationnaires, avec ou sans tension superficielle. Dans les deux cas, nous présentons une formulation lagrangienne naturelle, à partir de laquelle (différentes) formulations hamiltoniennes sont dérivées à l'aide de la théorie de la dualité, dans l'esprit de la transformée de Legendre–Fenchel. La démarche est systématique et clarifie une approche *ad hoc* récente de Kozlov et Kuznetsov [7].

1. Introduction

In *Hamiltonian spatial dynamics*, a physical problem is formulated as a Hamiltonian evolutionary equation in which an unbounded spatial direction plays the role of time. Rigorous spatial Hamiltonian formulations of the classical irrotational steady water-wave problem were given by Groves and Toland [4] (who clarified and unified previous results by Baesens and MacKay [1] and Mielke [8]) and Groves [3] (see also Benjamin [2, Appendix B]). In the first theory the hydrodynamic problem is formulated in terms of a velocity potential and the variable fluid domain mapped to a fixed strip by scaling the vertical coordinate by the free-surface elevation, while in the second it is formulated in terms of a stream function and the mapping to a fixed reference domain is achieved by means of a semi-hodograph transformation. The latter approach was extended to rotational water waves by Groves and Wahlén [5,6].

Kozlov and Kuznetsov [7] have recently examined the stream-function formulation of the hydrodynamic problem for rotational gravity waves using the scaling transformation and give variables whose role as coordinates in a Hamiltonian

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formulation are confirmed *a posteriori*. In this note we show that, contrary to a remark by Kozlov and Kuznetsov, their Hamiltonian system can be obtained by duality theory in the spirit of the Legendre–Fenchel transform from the corresponding Lagrangian system (which is readily derived from a variational principle). This systematic treatment, which is a straightforward modification of Groves and Toland's [4] theory, leads to (different) Hamiltonian formulations for both gravity and gravity–capillary waves and explains both the relationship between the two cases and Kozlov and Kuznetsov's choice of coordinates; in particular we explain how their manifolds M' and M'' emerge naturally (they correspond to N (Section 3) and $\mathcal{D}(v_{\mathrm{H}})$ (Theorem 4.3) in this note).

Consider a steady rotational flow with prescribed vorticity function $\omega \in C^{0,1}(\mathbb{R})$, total head R and volume flux Q. In terms of dimensionless variables (obtained by scaling lengths by $h_{\mathrm{c}} = g^{-\frac{1}{3}} Q^{\frac{2}{3}}$ and ψ by Q), the mathematical problem is to find a positive function η defining the fluid domain $D_\eta = \{(x,y)\colon\ x \in (x_1,x_2),\ y \in (0,\eta(x))\}$ and a stream function $\psi : \overline{D}_\eta \to \mathbb{R}$ which satisfies the equations

$$\begin{aligned}
&-\Delta\psi = \omega(\psi), && (x,y) \in D_\eta,\\
&\psi = 0, && y = 0,\\
&\psi = 1, && y = \eta(x),\\
&|\nabla\psi|^2 + 2\eta - 2\beta\left(\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right)_x = 3r, && y = \eta(x).
\end{aligned}$$

The dimensionless parameters β and r are given by $\beta = \sigma/(gh_{\mathrm{c}})$, $r = R/R_{\mathrm{c}}$, where σ is the coefficient of surface tension, g is the acceleration due to gravity and $R_{\mathrm{c}} = \frac{3}{2}gh_{\mathrm{c}}$, so that h_{c} and R_{c} are the critical depth and corresponding critical value of R for an irrotational flow with volume flux Q (e.g., see Benjamin [2]). Our starting point is the observation that these equations follow from the formal variational principle

$$\delta \int \left\{ \int_0^{\eta(x)} \left(\frac{1}{2}|\nabla\psi|^2 - \Omega(\psi)\right) \mathrm{d}y + \eta^2 + 2\beta\left(\sqrt{1+\eta_x^2} - 1\right) - 3r\eta \right\} \mathrm{d}x = 0, \tag{1}$$

where $\Omega(t) := -\int_t^1 \omega(s)\,\mathrm{d}s$ and the variations are taken in η and ψ and with compact support in (a,b), $a < x_1 < x_2 < b$.

2. Lagrangian formulation

Consider the change of variable $z = yh/\eta(x)$, $\Phi(x,z) = \psi(x,y)$, where h is a positive constant (e.g. select h such that $(\eta,\psi) = (h,\psi_h(y))$ is a horizontal laminar flow, that is $\psi_h'' = \omega(\psi_h)$ for $y \in (0,h)$ with $\psi_h(0) = 0$, $\psi_h(h) = 1$ and $\psi'(h)^2 + 2h = r$). This transformation converts the hydrodynamic problem into

$$\Phi_{xx} - \frac{2z\eta_x}{\eta}\Phi_{xz} + \frac{2z\eta_x^2}{\eta^2}\Phi_z + \frac{z^2\eta_x^2}{\eta^2}\Phi_{zz} - \frac{z\eta_{xx}}{\eta}\Phi_z + \frac{h^2}{\eta^2}\Phi_{zz} - \omega(\Phi) = 0, \qquad 0 < z < h, \tag{2}$$

$$\Phi = 0, \qquad z = 0, \tag{3}$$

$$\Phi = 1, \qquad z = h, \tag{4}$$

$$\left(\Phi_x - \frac{\eta_x}{\eta}\Phi_z\right)^2 + \frac{h^2\Phi_z^2}{\eta^2} + 2\eta - 2\beta\left[\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right]_x - 3r = 0, \qquad z = h, \tag{5}$$

where $x \in (x_1,x_2)$, and the variational principle (1) into

$$\delta \int L(\eta,\Phi;\eta_x,\Phi_x)\,\mathrm{d}x = 0,$$

where

$$L(\eta,\Phi;\mu,\Psi) := \frac{1}{h\eta}\int_0^h \left\{\frac{1}{2}(\eta\Psi - \mu z\Phi_z)^2 + \frac{1}{2}h^2\Phi_z^2 - \eta^2\Omega(\Phi)\right\}\mathrm{d}z + \eta^2 + 2\beta\left(\sqrt{1+\mu^2} - 1\right) - 3r\eta. \tag{6}$$

Observe that

$$Q := \left\{(\eta,\Phi) \in (0,\infty) \times H^1(0,h)\colon\ \Phi(0) = 0,\ \Phi(h) = 1\right\} \tag{7}$$

is a Hilbert manifold modelled on the single coordinate chart $(0,\infty) \times H_0^1(0,h)$; one can take $(\eta,\psi) \mapsto (\eta,\psi+\psi_h)$ as a coordinate map $(0,\infty) \times H_0^1(0,h) \to Q$ with a concrete physical interpretation (Φ is written as a perturbation of a laminar flow). Eq. (6) defines a Lagrangian $L \in C^1(TQ, \mathbb{R})$, where $TQ = \bigcup_{p\in Q}\{p\} \times TQ|_p \cong Q \times \mathbb{R} \times H_0^1(0,h)$ is the tangent bundle of Q.

The following result is proved by the arguments given by Groves and Toland [4, pp. 220–221].

Lemma 2.1. *Define $X = \{\gamma \in C^1([a,b], Q)\colon\ \gamma(a) = q_1,\ \gamma(b) = q_2\}$ for fixed $q_1, q_2 \in Q$ and $\mathcal{L} \in C^1(X, \mathbb{R})$ by*

$$\mathcal{L}(\gamma) = \int_a^b L\bigl(\gamma(x); \dot{\gamma}(x)\bigr)\,\mathrm{d}x.$$

Suppose that the twice continuously differentiable path $\gamma = (\eta, \Phi) : [a,b] \to Q$ satisfies Lagrange's equation

$$\mathbf{d}\mathcal{L}|_\gamma(\gamma_1) = \int_a^b \bigl\{\mathrm{d}_1 L\bigl[\gamma(x); \dot{\gamma}(x)\bigr]\bigl(\gamma_1(x)\bigr) + \mathrm{d}_2 L\bigl[\gamma(x); \dot{\gamma}(x)\bigr]\bigl(\dot{\gamma}_1(x)\bigr)\bigr\}\,\mathrm{d}x = 0$$

for all $\gamma_1 \in TX|_\gamma \cong C_0^1([a,b], \mathbb{R} \times H_0^1(0,h))$. There exists a measurable function $\tilde{\Phi}(x,z)$, $(x,z) \in (a,b) \times (0,h)$, with the properties that for all $x \in (a,b)$ the equation $\tilde{\Phi}(x,z) = \Phi(x)(z)$ holds for almost all $z \in [0,h]$. Furthermore, $\tilde{\Phi} \in H^2[(x_1,x_2) \times (0,h)]$ and is a strong solution of (2)–(5) for $a < x_1 < x_2 < b$.

3. Legendre–Fenchel transform

The following construction is motivated by classical duality theory but avoids any assumptions on the behaviour of L. Consider a manifold domain Q of a manifold P and a Lagrangian $L \in C^1(\tilde{T}Q, \mathbb{R})$, where $\tilde{T}Q = \bigcup_{q\in Q}\{q\} \times TP|_q$ (a manifold domain of the tangent bundle TP of P). Define the Hamiltonian $H : \mathcal{D}(H) \subseteq \tilde{T}^*Q \to \mathbb{R}$, where $\tilde{T}^*Q = \bigcup_{q\in Q}\{q\} \times T^*P|_q$ (a manifold domain of the cotangent bundle T^*P of P) by the formula:

$$H\bigl(q, p^*\bigr) := \sup\bigl\{p^*(p) - L(q,p)\colon\ p \in TP|_q\bigr\},$$

where the domain $\mathcal{D}(H)$ of H is the interior of the set for which the supremum is finite, and equip this set (which is an immersed submanifold of T^*P) with the restriction of the canonical 2-form Ω_c on T^*P.

To apply this construction to steady water waves, let us define Q by Eq. (7), so that it is a manifold domain of $P = (0,\infty) \times L^2(0,h)$ (which is a Hilbert manifold defined on the single coordinate chart $\mathbb{R} \times L^2(0,h)$), and L by Eq. (6), so that $L \in C^1(\tilde{T}Q, \mathbb{R})$, where $\tilde{T}Q \cong Q \times \mathbb{R} \times L^2(0,h)$. The Hamiltonian $H : \mathcal{D}(H) \subseteq \tilde{T}^*Q \to \mathbb{R}$ is given by the formula

$$\begin{aligned} H(\eta, \Phi, w, \Psi) &:= \sup\left\{ w\mu + \int_0^h \Psi\theta\,\mathrm{d}z - L(\eta, \Phi; \mu, \theta)\colon\ (\mu,\theta) \in \mathbb{R} \times L^2(0,h)\right\} \\ &= \sup_{\mu\in\mathbb{R}}\left\{ w\mu - 2\beta\sqrt{1+\mu^2} + \sup_{\theta\in L^2(0,h)} \int_0^h \left(\Psi\theta - \frac{1}{2h\eta}(\eta\theta - \mu z\Phi_z)^2\right)\mathrm{d}z\right\} \\ &\quad + \int_0^h \left(\frac{\eta}{h}\Omega(\Phi) - \frac{h}{2\eta}\Phi_z^2\right)\mathrm{d}z - \eta^2 + 2\beta + 3r\eta; \end{aligned} \tag{8}$$

here, and in the following calculations, we use the coordinates $\tilde{T}^*Q \cong Q \times \mathbb{R} \times L^2(0,h)$. For each fixed $\mu \in \mathbb{R}$, the supremum

$$\sup_{\theta\in L^2(0,h)} \int_0^h \left(\Psi\theta - \frac{1}{2h\eta}(\eta\theta - \mu z\Phi_z)^2\right)\mathrm{d}z$$

is finite and attained when the integrand is maximized for each $z \in (0,h)$, that is, when $\theta = \theta_{\max} := \frac{1}{\eta}(h\Psi + \mu z\Phi_z) \in L^2(0,h)$. Evaluating the supremum at $\theta = \theta_{\max}$, we find from (8) that

$$H(\eta, \Phi, w, \Psi) = \int_0^h \left(\frac{h}{2\eta}\bigl(\Psi^2 - \Phi_z^2\bigr) + \frac{\eta}{h}\Omega(\Phi)\right)\mathrm{d}z - \eta^2 + 2\beta + 3r\eta + \sup_{\mu\in\mathbb{R}}\bigl\{W\mu - 2\beta\sqrt{1+\mu^2}\bigr\},$$

where $W := w + \frac{1}{\eta}\int_0^h z\Psi\Phi_z\,\mathrm{d}z$. For each fixed $W \in \mathbb{R}$ the function $\mu \mapsto W\mu - 2\beta\sqrt{1+\mu^2}$ is bounded above if and only if $|W| \leqslant 2\beta$, in which case its supremum is $-\sqrt{4\beta^2 - W^2}$, so that

$$H(\eta,\Phi,w,\Psi)=\int_0^h\left(\frac{h}{2\eta}\left(\Psi^2-\Phi_z^2\right)+\frac{\eta}{h}\Omega(\Phi)\right)\mathrm{d}z-\eta^2+2\beta+3R\eta-\sqrt{4\beta^2-W^2}, \tag{9}$$

with

$$\mathcal{D}(H)=M:=\left\{(\eta,\Phi,w,\Psi)\in Q\times\mathbb{R}\times L^2(0,h):\ |W|<2\beta\right\}$$

for $b>0$ and

$$\mathcal{D}(H)=N:=\left\{(\eta,\Phi,w,\Psi)\in Q\times\mathbb{R}\times L^2(0,h):\ W=0\right\}=\left\{\left(\eta,\Phi,F(\eta,\Phi,\Psi),\Psi\right)\in P\times L^2(0,h)\right\}$$

for $b=0$, where $F(\eta,\Phi,\Psi):=-\frac{1}{\eta}\int_0^h z\Phi_z\Psi\,\mathrm{d}z$.

Finally, we define Hamiltonian systems (M,Ω_M,H) and (N,Ω_N,H) by equipping M and N with the restrictions Ω_M and Ω_N of the (position-independent) canonical 2-form Ω_c on T^*P, which is defined by the formula

$$\Omega_\mathrm{c}\left((\eta_1,\Phi_1,w_1,\Psi_1),(\eta_2,\Phi_2,w_2,\Psi_2)\right)=w_2\eta_1-w_1\eta_2+\int_0^h(\Psi_2\Phi_1-\Psi_1\Phi_2)\,\mathrm{d}z. \tag{10}$$

Clearly M is a manifold domain of T^*P modelled upon the single coordinate chart $(0,\infty)\times H_0^1(0,h)\times\mathbb{R}\times L^2(0,h)$, H and Ω_M are given by (9) and (10) in this coordinate system and (M,Ω_M) is a symplectic manifold. Furthermore N is an embedded submanifold of M modelled upon the single coordinate chart $(0,\infty)\times H_0^1(0,h)\times L^2(0,h)$; the coordinate map $(0,\infty)\times H_0^1(0,h)\times L^2(0,h)\to N$ is given by $(\eta,\psi,\Psi)\mapsto(\eta,\Phi,F(\eta,\Phi,\Psi),\Psi)$, where $(\eta,\psi)\mapsto(\eta,\Phi)$ is the coordinate map $(0,\infty)\times H_0^1(0,h)\to Q$. A straightforward calculation shows that H and Ω_N are given in this coordinate system by (9) and

$$\begin{aligned}\Omega_N|_{(\eta,\Phi,\Psi)}\left((\eta_1,\Phi_1,\Psi_1),(\eta_2,\Phi_2,\Psi_2)\right)=&-\frac{\eta_1}{\eta}\int_0^h z(\Phi_{2z}\Psi+\Phi_z\Psi_2)\,\mathrm{d}z\\&+\frac{\eta_2}{\eta}\int_0^h z(\Phi_{1z}\Psi+\Phi_z\Psi_1)\,\mathrm{d}z+\int_0^h(\Psi_2\Phi_1-\Phi_2\Psi_1)\,\mathrm{d}z.\end{aligned}$$

Note that Ω_N is closed but not weakly nondegenerate at all points of N (see below).

4. Hamiltonian formulation

Finally, we compute Hamilton's equations for the Hamiltonian systems (M,Ω_M,H) and (N,Ω_N,H) and examine their connection to the water-wave problem.

Theorem 4.1. *The domain of the Hamiltonian vector field v_H corresponding to the Hamiltonian system (M,Ω_M,H) is the set*

$$\mathcal{D}(v_\mathrm{H})=\left\{(\eta,\Phi,w,\Psi)\in M:\ (\Phi,\Psi)\in H^2(0,h)\times H^1(0,h)\ \text{with}\ \Psi(0)=0,\ \Psi(h)=\frac{W}{\sqrt{4\beta^2-W^2}}\Phi_z(h)\right\}$$

and v_H is given by the mapping

$$\begin{pmatrix}\eta\\ \Phi\\ w\\ \Psi\end{pmatrix}\mapsto\begin{pmatrix}\frac{W}{\sqrt{4\beta^2-W^2}}\\ \frac{h}{\eta}\Psi+\frac{zW\Phi_z}{\eta\sqrt{4\beta^2-W^2}}\\ \frac{h}{2\eta^2}\int_0^h(\Psi^2-\Phi_z^2)\,\mathrm{d}z+\frac{W}{\eta^2\sqrt{4\beta^2-W^2}}\int_0^h z\Psi\Phi_z\,\mathrm{d}z-\frac{1}{h}\int_0^h\Omega(\Phi)\,\mathrm{d}z+2\eta-3r\\ -\frac{h}{\eta}\Phi_{zz}+\frac{W}{\eta\sqrt{4\beta^2-W^2}}(z\Psi)_z-\frac{\eta}{h}\omega(\Phi)\end{pmatrix}.$$

Proof. The point $(\eta,\Phi,w,\Psi)\in M$ belongs to the domain of v_H with $v_\mathrm{H}(\eta,\Phi,w,\Psi)=(\overline{\eta},\overline{\Phi},\overline{w},\overline{\Psi})\in TM|_{(\eta,\Phi,w,\Psi)}$ if and only if

$$\Omega_M\left((\overline{\eta},\overline{\Phi},\overline{w},\overline{\Psi}),(\hat{\eta},\hat{\Phi},\hat{w},\hat{\Psi})\right)=\mathbf{d}H|_{(\eta,\Phi,w,\Psi)}(\hat{\eta},\hat{\Phi},\hat{w},\hat{\Psi})$$

for all $(\hat{\eta},\hat{\Phi},\hat{w},\hat{\Psi})\in TM|_{(\eta,\Phi,w,\Psi)}\cong\mathbb{R}\times H_0^1(0,h)\times\mathbb{R}\times L^2(0,h)$, that is,

$$\begin{aligned}\bar{\eta}\hat{w} - \bar{w}\hat{\eta} + \int_0^h (\bar{\Phi}\hat{\Psi} - \bar{\Psi}\hat{\Phi})\,\mathrm{d}z = {} & -\frac{h\hat{\eta}}{2\eta^2}\int_0^h \left(\Psi^2 - \Phi_z^2\right)\mathrm{d}z + \frac{h}{\eta}\int_0^h (\Psi\hat{\Psi} - \Phi_z\hat{\Phi}_z)\,\mathrm{d}z \\ & + \frac{\hat{\eta}}{h}\int_0^h \Omega(\Phi)\,\mathrm{d}z + \frac{\eta}{h}\int_0^h \omega(\Phi)\hat{\Phi}\,\mathrm{d}z \\ & + \frac{W}{\sqrt{4\beta^2 - W^2}}\left(\hat{w} - \frac{\hat{\eta}}{\eta^2}\int_0^h z\Psi\Phi_z\,\mathrm{d}z + \frac{1}{\eta}\int_0^h (\hat{\Psi}\Phi_z + \Psi\hat{\Phi}_z)z\,\mathrm{d}z\right) - 2\eta\hat{\eta} + 3r\hat{\eta}.\end{aligned} \tag{11}$$

Setting $\hat{\eta}, \hat{\Phi}, \hat{\Psi}$ and $\hat{w}, \hat{\Phi}, \hat{\Psi}$ equal to zero, we find that $\bar{\eta}$ and $\bar{w}$ are given by the right-hand sides of the first and third components of the given formula for v_{H}, and substituting these formulae into (11) yields:

$$\int_0^h (\bar{\Phi}\hat{\Psi} - \bar{\Psi}\hat{\Phi})\,\mathrm{d}z = \frac{h}{\eta}\int_0^h (\Psi\hat{\Psi} - \Phi_z\hat{\Phi}_z)\,\mathrm{d}z + \frac{W}{\eta\sqrt{4\beta^2 - W^2}}\int_0^h (\hat{\Psi}\Phi_z + \Psi\hat{\Phi}_z)z\,\mathrm{d}z + \frac{\eta}{h}\int_0^h \omega(\Phi)\hat{\Phi}\,\mathrm{d}z.$$

Setting $\hat{\Phi} = 0$, we therefore find that

$$\int_0^h \left(\frac{h}{\eta}\Psi + \frac{zW\Phi_z}{\eta\sqrt{4\beta^2 - W^2}} - \bar{\Phi}\right)\hat{\Psi}\,\mathrm{d}z = 0$$

for all $\hat{\Psi} \in L^2(0,h)$, so that

$$\bar{\Phi} = \frac{h}{\eta}\Psi + \frac{zW\Phi_z}{\eta\sqrt{4\beta^2 - W^2}} \in H_0^1(0,h). \tag{12}$$

On the other hand, setting $\hat{\Psi} = 0$ yields

$$\int_0^h \left\{\left(\bar{\Psi} + \frac{\eta}{h}\omega(\Phi)\right)\hat{\Phi} + \hat{\Phi}_z\left(-\frac{h}{\eta}\Phi_z + \frac{zW\Psi}{\eta\sqrt{4\beta^2 - W^2}}\right)\right\}\mathrm{d}z = 0$$

for all $\hat{\Phi} \in H_0^1(0,h)$, so that

$$\left(-\frac{h}{\eta}\Phi_z + \frac{zW\Psi}{\eta\sqrt{4\beta^2 - W^2}}\right)_z = \bar{\Psi} + \frac{\eta}{h}\omega(\Phi) \in L^2(0,h) \tag{13}$$

($\Phi \in H^1(0,h) \subseteq C[0,h]$ and $\Omega \in C(\mathbb{R})$, so that $\Omega(\Phi) \in C[0,h] \subset L^2(0,h)$). It follows in particular from Eqs. (12) and (13) that $\Phi \in H^2(0,h)$, $\Psi \in H^1(0,h)$, and evaluating (12) at $z = 0$ and $z = h$, one obtains the boundary conditions for Ψ given in the definition of $\mathcal{D}(v_{\mathrm{H}})$. □

Details of the arguments needed to prove the next theorem are given by Groves and Toland [4, pp. 212–213]. Eliminating w and $\tilde{\Psi}$ from the equations in the theorem, we find that $\tilde{\Phi}$ is a strong solution of (2)–(5) in $(x_1, x_2) \times (0,h)$ for $a < x_1 < x_2 < b$.

Theorem 4.2. *Suppose that the continuously differentiable path $\gamma = (\eta, \Phi, w, \Psi) : (a,b) \to M$ satisfies Hamilton's equations for (M, Ω_M, H). There exist measurable functions $\tilde{\Phi}(x,z)$ and $\Psi(x,z)$, $(x,z) \in (a,b) \times (0,h)$, with the properties that for all $x \in (a,b)$ the equations $\tilde{\Phi}(x,z) = \Phi(x)(z)$, $\tilde{\Psi}(x,z) = \Psi(x)(z)$ hold for almost all $z \in [0,h]$. Furthermore, η and w are twice continuously differentiable in (x_1, x_2) and $\tilde{\Phi} \in H^2[(x_1,x_2) \times (0,h)]$, $\tilde{\Psi} \in H^1[(x_1,x_2) \times (0,h)]$ for $a < x_1 < x_2 < b$ with*

$$\eta_x = \frac{\tilde{W}}{\sqrt{4\beta^2 - \tilde{W}^2}}, \qquad w_x = \frac{h}{2\eta^2}\int_0^h \left(\tilde{\Psi}^2 - \tilde{\Phi}_z^2\right)\mathrm{d}z + \frac{\tilde{W}}{\eta^2\sqrt{4\beta^2 - \tilde{W}^2}}\int_0^h z\tilde{\Psi}\tilde{\Phi}_z\,\mathrm{d}z - \frac{1}{h}\int_0^h \Omega(\tilde{\Phi})\,\mathrm{d}z + 2\eta - 3r$$

and

$$\tilde{\Phi}_x = \frac{h}{\eta}\tilde{\Psi} + \frac{z\tilde{W}\tilde{\Phi}_z}{\eta\sqrt{4\beta^2 - \tilde{W}^2}}, \qquad \tilde{\Psi}_x = -\frac{h}{\eta}\tilde{\Phi}_{zz} + \frac{\tilde{W}}{\eta\sqrt{4\beta^2 - \tilde{W}^2}}(z\tilde{\Psi})_z - \frac{\eta}{h}\omega(\tilde{\Phi})$$

with boundary conditions

$$\tilde{\Phi}(0)=0, \qquad \tilde{\Psi}(0)=0, \qquad \tilde{\Phi}(h)=1, \qquad \tilde{\Psi}(h)=\frac{\tilde{W}}{\sqrt{4\beta^2-\tilde{W}^2}}\tilde{\Phi}_z(h),$$

where $\tilde{W}=w+\frac{1}{\eta}\int_0^h z\tilde{\Psi}\tilde{\Phi}_z\,\mathrm{d}z$ *satisfies* $|\tilde{W}|<2\beta$.

Turning to (N,Ω_N,H), recall that Ω_N is not weakly nondegenerate at all points of N, so that the Hamiltonian vector field is not uniquely defined at all points of its domain; one proceeds by allowing it to associate a set T of tangent vectors in the space $TN|_n$ with each point (η,Φ,Ψ) of $\mathcal{D}(v_{\mathrm{H}})$ and defining its *essential domain* $\mathcal{D}_{\mathrm{ess}}(v_{\mathrm{H}})$ as the subset of elements $(\eta,\Phi,\Psi)\in\mathcal{D}(v_{\mathrm{H}})$ for which $v|_{(\eta,\Phi,\Psi)}$ is a single tangent vector.

Theorem 4.3. *Let* v_{H} *denote the Hamiltonian vector field corresponding to the Hamiltonian system* (N,Ω_N,H).

(1) *The domain of* v_{H} *is the set*

$$\mathcal{D}(v_{\mathrm{H}})=\bigg\{(\eta,\Phi,\Psi)\in N\colon\ (\Phi,\Psi)\in H^2(0,h)\times H^1(0,h)\ \textit{with}\ \text{(i)}\ \Psi(0)=0,$$
$$\text{(ii)}\ \frac{h^2}{2\eta^2}\big(\Psi^2(h)+\Phi_z^2(h)\big)=-2\eta+3r,\ \text{(iii)}\ \Phi_z(h)=0\ \textit{implies}\ \Psi(h)=0\bigg\}.$$

(2) *For each* $(\eta,\Phi,\Psi)\in N$ *the following statements are equivalent*
 (i) $(\eta,\Phi,\Psi)\in\mathcal{D}_{\mathrm{ess}}(v_{\mathrm{H}})$;
 (ii) (η,Φ,Ψ) *is not a* ***surface stagnation point*** *of* N, *that is,* $\Phi_z(h)^2+\Psi(h)^2\neq 0$;
 (iii) $\Phi_z(h)\neq 0$.

(3) *For each* $(\eta,\Phi,\Psi)\in\mathcal{D}_{\mathrm{ess}}(v_{\mathrm{H}})$, *the Hamiltonian vector field is given by*

$$v_{\mathrm{H}}(\eta,\Phi,\Psi)=\left(-\frac{\Psi(h)}{\Phi_z(h)},\frac{h}{\eta}\Psi-\frac{\Psi(h)}{\Phi_z(h)\eta}z\Phi_z,-\frac{h}{s}\Phi_{zz}-\frac{\Psi(h)}{\Phi_z(h)\eta}(z\Psi)_z-\frac{\eta}{h}\omega(\Phi)\right).$$

(4) *For each surface stagnation point* $(\eta,\Phi,\Psi)\in\mathcal{D}(v_{\mathrm{H}})\setminus\mathcal{D}_{\mathrm{ess}}(v_{\mathrm{H}})$ *the Hamiltonian vector field is the set*

$$v_{\mathrm{H}}(\eta,\Phi,\Psi)=\left\{\left(\bar{\eta},\frac{h}{\eta}\Psi+\frac{\bar{\eta}}{\eta}z\Phi_z,-\frac{h}{\eta}\Phi_{zz}+\frac{\bar{\eta}}{\eta}(z\Psi)_z-\frac{\eta}{h}\omega(\Phi)\right)\right\}_{\bar{\eta}\in\mathbb{R}}.$$

Proof. The point $(\eta,\Phi,\Psi)\in M$ belongs to the domain of v_{H} with $v_{\mathrm{H}}(\eta,\Phi,w,\Psi)=(\bar{\eta},\overline{\Phi},\overline{\Psi})\in TM|_{(\eta,\Phi,\Psi)}$ if and only if

$$\Omega_N|_{(\eta,\Phi,\Psi)}\big((\bar{\eta},\overline{\Phi},\overline{\Psi}),(\hat{\eta},\hat{\Phi},\hat{\Psi})\big)=\mathbf{d}H|_{(\eta,\Phi,\Psi)}(\hat{\eta},\hat{\Phi},\hat{\Psi})$$

for all $(\hat{\eta},\hat{\Phi},\hat{\Psi})\in TM|_{(\eta,\Phi,\Psi)}\cong\mathbb{R}\times H_0^1(0,h)\times L^2(0,h)$, that is,

$$\begin{aligned}&-\frac{\bar{\eta}}{\eta}\int_0^h(\hat{\Phi}_z\Psi+\Phi_z\hat{\Psi})z\,\mathrm{d}z+\frac{\hat{\eta}}{\eta}\int_0^h(\bar{\Phi}_z\Psi+\Phi_z\bar{\Psi})z\,\mathrm{d}z+\int_0^h(\bar{\Phi}\hat{\Psi}-\bar{\Psi}\hat{\Phi})\,\mathrm{d}z\\&\quad=-\frac{h\hat{\eta}}{2\eta^2}\int_0^h\big(\Psi^2-\Phi_z^2\big)\,\mathrm{d}z+\frac{h}{\eta}\int_0^h(\Psi\hat{\Psi}-\Phi_z\hat{\Phi}_z)\,\mathrm{d}z+\frac{\hat{\eta}}{h}\int_0^h\Omega(\Phi)\,\mathrm{d}z+\frac{\eta}{h}\int_0^h\omega(\Phi)\hat{\Phi}\,\mathrm{d}z-2\eta\hat{\eta}+3r\hat{\eta}.\end{aligned}\tag{14}$$

Setting $(\hat{\Phi},\hat{\Psi})$ and $(\hat{\eta},\hat{\Phi})$ equal to zero, we find that

$$\frac{1}{\eta}\int_0^h(\bar{\Phi}_z\Psi+\Phi_z\bar{\Psi})z\,\mathrm{d}z=-\frac{h}{2\eta^2}\int_0^h\big(\Psi^2-\Phi_z^2\big)\,\mathrm{d}z+\frac{1}{h}\int_0^h\Omega(\Phi)\,\mathrm{d}z-2\eta+3r\tag{15}$$

and

$$\int_0^h\left(\frac{h}{\eta}\Psi+\frac{\bar{\eta}}{\eta}z\Phi_z-\bar{\Phi}\right)\hat{\Psi}\,\mathrm{d}z=0,$$

and substituting these formulae into (14) yields

$$\int_0^h \left\{ \left(\bar{\Psi} + \frac{\eta}{h}\omega(\Phi) \right)\hat{\Phi} + \hat{\Phi}_z\left(-\frac{h}{\eta}\Phi_z + \frac{\bar{\eta}}{\eta}z\Psi \right) \right\} \mathrm{d}z = 0.$$

The argument used in the proof of Theorem 4.1 shows that $\Phi \in H^2(0,h)$ and $\Psi \in H^1(0,h)$, with

$$\bar{\Phi} = \frac{h}{\eta}\Psi + \frac{\bar{\eta}}{\eta}z\Phi_z, \qquad \bar{\Psi} = -\frac{h}{\eta}\Phi_{zz} + \frac{\bar{\eta}}{\eta}(z\Psi)_z - \frac{\eta}{h}\omega(\Phi) \tag{16}$$

and

$$\Psi(0) = 0, \qquad \Psi(h) = -\bar{\eta}\Phi_z(h), \tag{17}$$

so that $\bar{\eta} = -\Psi(h)/\Phi_z(h)$ if $\Phi_z(h) \neq 0$ and $\Psi(h) = 0$ if $\Phi_z(h) = 0$. It follows from (16) and (17) by a straightforward integration by parts that

$$\int_0^h (\bar{\Phi}_z\Psi + \Phi_z\bar{\Psi})z\,\mathrm{d}z = \frac{h^2}{2\eta}\left(\Psi^2(h) + \Phi_z^2(h)\right) - \frac{\eta}{h}\Omega\left(\Phi(h)\right) - \frac{h}{2\eta}\int_0^h \left(\Psi^2 - \Phi_z^2\right)\mathrm{d}z + \frac{\eta}{h}\int_0^h \Omega(\Phi)\,\mathrm{d}z,$$

and combining this equation with (15), one obtains condition (ii) in the definition of $\mathcal{D}(v_{\mathrm{H}})$ because $\Omega(\Phi(h)) = \Omega(1) = 0$.

This calculation shows that the conditions specified in part (1) of the theorem hold if (η, Φ, Ψ) belongs to $\mathcal{D}(v_{\mathrm{H}})$. Conversely, suppose that (η, Φ, Ψ) satisfies these conditions. The equation $\Psi(h) = -\bar{\eta}\Phi_z(h)$ has at least one solution for $\bar{\eta}$ because $\Psi(h) = 0$ implies that $\Phi_z(h) = 0$. With this choice of $\bar{\eta}$, define $\bar{\Phi}$, $\bar{\Psi}$ by (16). It is a straightforward matter to verify that $(\eta, \Phi, \Psi) \in \mathcal{D}(v_{\mathrm{H}})$ and $(\bar{\eta}, \bar{\Phi}, \bar{\Psi}) \in v_{\mathrm{H}}|_{(\eta,\Phi,\Psi)}$.

Suppose that (2)(i) is true, so that (η, Φ, Ψ) belongs to $\mathcal{D}_{\mathrm{ess}}(v_{\mathrm{H}})$. The properties specified in part (1) hold and $v_{\mathrm{H}}|_{(\eta,\Phi,\Psi)}$ is a single point; in particular, $\bar{\eta}$ is unique. It follows that $\Psi(h)$ is non-zero, so that (η, Φ, Ψ) is not a surface stagnation point. Next suppose that (2)(ii) is true, so that $(\eta, \Phi, \Psi) \in \mathcal{D}(v_{\mathrm{H}})$ is not a surface stagnation point. By definition, at least one of $\Psi(h)$ and $\Phi_z(h)$ is non-zero. Part (1) of the theorem implies that $\Psi(h)$ is non-zero. Finally, suppose that (2)(iii) is true. The equation $\Psi(h) = -\bar{\eta}\Phi_z(h)$ has a unique solution $\bar{\eta}$ which may be used to define a unique $\bar{\Phi}$ and $\bar{\Psi}$ by (16). Since $(\bar{\eta}, \bar{\Phi}, \bar{\Psi})$ is unique, it follows that $(\eta, \Phi, \Psi) \in \mathcal{D}_{\mathrm{ess}}(v_{\mathrm{H}})$.

The Hamiltonian vector field $v_{\mathrm{H}}|_n$ at any point $n \in \mathcal{D}(v_{\mathrm{H}})$ is found by solving $\Psi(h) = -\bar{\eta}\Phi_z(h)$ for $\bar{\eta}$ (the solution is unique and equal to $\Phi_z(h)/\Psi(h)$ for $(\eta, \Phi, \Psi) \in \mathcal{D}_{\mathrm{ess}}(v_{\mathrm{H}})$; otherwise $\bar{\eta}$ can take any value in $\mathbb{R}$) and defining $\bar{\Phi}$, $\bar{\Psi}$ by (16). □

A continuously differentiable path $\gamma : (a,b) \to M$ with $\gamma(x) = (\eta(x), \Phi(x), \Psi(x)) \in \mathcal{D}_{\mathrm{ess}}(v_{\mathrm{H}})$ for all $x \in (a,b)$ leads to a solution of the water-wave equations (2)–(5) using the procedure explained in Theorem 4.2. (At a surface stagnation point $|\nabla\psi(h)|^2 = (\Phi_x(h) - \frac{\eta_x}{\eta}\Phi_z(h))^2 + \frac{h^2}{\eta^2}\Phi_z(h)^2 = 0$, and η attains its maximum value of $\frac{3}{2}r$; at such a point either $\eta_x = 0$ or η_x has a singularity. This matter is discussed in detail by Varvaruca and Weiss [9].)

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