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Complex analysis

Coefficient estimates for certain classes of meromorphic bi-univalent functions

Estimations des coefficients pour certaines classes de fonctions méromorphes bi-univalentes

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ABSTRACT

We define a new class of meromorphic bi-univalent functions and use the Faber polynomial expansions to determine the coefficient bounds for such functions. Our results generalize and/or improve some of the previously known results. A meromorphic function is said to be bi-univalent in a given domain $\triangle$ if both the function and its inverse map are univalent there.

RÉSUMÉ

Nous définissons une nouvelle classe de fonctions méromorphes bi-univalentes et utilisons des développements en polynômes de Faber pour déterminer des bornes sur les coefficients de ces fonctions. Nos résultats généralisent et améliorent certains résultats antérieurement connus. Une fonction méromorphe est dite ici être bi-univalente dans un domaine donné $\triangle$ si la fonction et sa fonction réciproques y sont toutes deux univalentes.

1. Introduction

Let Σ denote the class of meromorphic univalent functions g of the form

$$g(z) = z + b_0 + \sum_{n=1}^{\infty} \frac{b_n}{z^n} \tag{1.1}$$

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defined on the domain $\Delta = \{z\colon\ 1 < |z| < \infty\}$. Since $g \in \Sigma$ is univalent, it has an inverse $g^{-1} = h$, that satisfy

$$g^{-1}\big(g(z)\big) = z, \quad (z \in \Delta)$$

and

$$g\big(g^{-1}(w)\big) = w, \quad \big(M < |w| < \infty, M > 0\big).$$

The inverse function $h = g^{-1}$ has a series expansion of the form (e.g. see [1], [2] or [13]):

$$h(w) = g^{-1}(w) = w + \sum_{n=0}^{\infty} \frac{B_n}{w^n}, \quad (M < |w| < \infty). \tag{1.2}$$

A function $g \in \Sigma$ is said to be meromorphic bi-univalent if the inverse map $h = g^{-1}$ is also in Σ.

Estimates on the coefficients of the inverses of meromorphic univalent functions were widely investigated in the literature. For example, Schiffer [8] obtained the estimate $|b_2| \leqslant \frac{2}{3}$ for meromorphic univalent functions $g \in \Sigma$ with $b_0 = 0$ and Duren [3] proved that $|b_n| \leqslant \frac{2}{(n+1)}$ if $b_k = 0$ for $1 \leqslant k < \frac{n}{2}$. For the coefficients of inverses of meromorphic univalent functions, Springer [11] proved that $|B_3| \leqslant 1$ and $|B_3 + \frac{1}{2}B_1^2| \leqslant \frac{1}{2}$ and conjectured that $|B_{2n-1}| \leqslant \frac{(2n-1)!}{n!(n-1)!}$, $(n = 1, 2, \ldots)$. Kubota [7] proved that the Springer conjecture is true for $n = 3, 4, 5$, and subsequently Schober [10] obtained sharp bounds for $|B_{2n-1}|$ if $1 \leqslant n \leqslant 7$.

In 2007, Kapoor and Mishra [6] found the coefficient estimates for the inverse of meromorphic starlike univalent functions of order α in Δ. In 2011, Srivastava et al. [12] found sharp bounds for the coefficients of inverses of starlike univalent functions of order α $(0 \leqslant \alpha < 1)$ having m-fold gap series representation. Recently, Hamidi et al. [4,5] used Faber polynomial expansions to find coefficient bounds for two classes of meromorphic bi-starlike and bi-univalent functions.

Motivated by the work of Hamidi et al. [4,5], we use Faber polynomial expansions for a larger subclass of meromorphic bi-univalent functions which includes the two classes of bi-univalent functions studied in [4,5]. For this class, the coefficient bounds for $|b_n|$ are determined. We also introduce bounds for $|b_0|$ to demonstrate the unpredictability of the coefficients of bi-univalent functions. As a special case, we also improve a coefficient bound presented in [4].

2. The class $\mathcal{M}_\Sigma(\lambda, \mu, \alpha)$

First, we define a comprehensive class of meromorphic bi-univalent functions which includes the two classes of bi-univalent functions studied in [4,5].

Definition 2.1. For $0 \leqslant \lambda \leqslant 1$ and $\mu \geqslant 0$, a function $g \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{M}_\Sigma(\lambda, \mu, \alpha)$ if the following conditions are satisfied:

$$\Re\left((1-\lambda)\left(\frac{g(z)}{z}\right)^{\mu} + \lambda g'(z)\left(\frac{g(z)}{z}\right)^{\mu-1}\right) > \alpha$$

and

$$\Re\left((1-\lambda)\left(\frac{h(w)}{w}\right)^{\mu} + \lambda h'(w)\left(\frac{h(w)}{w}\right)^{\mu-1}\right) > \alpha$$

where $0 \leqslant \alpha < 1$ and $z, w \in \Delta$ and the function $h = g^{-1}$ is given by (1.2).

In the following examples, we show how the class of meromorphic bi-univalent functions $\mathcal{M}_\Sigma(\lambda, \mu, \alpha)$ for suitable choices of λ and μ lead to certain new as well as known classes of meromorphic bi-univalent functions studied earlier in the literature.

Example 2.1. For $0 \leqslant \lambda \leqslant 1$ and $\mu = 1$, a function $g \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{M}_\Sigma(\lambda, 1, \alpha) \equiv \mathcal{F}_\Sigma(\lambda, \alpha)$ if it satisfies the following conditions:

$$\Re\left((1-\lambda)\frac{g(z)}{z} + \lambda g'(z)\right) > \alpha \quad \text{and} \quad \Re\left((1-\lambda)\frac{h(w)}{w} + \lambda h'(w)\right) > \alpha$$

where $0 \leqslant \alpha < 1$ and $z, w \in \Delta$ and the function h is given by (1.2).

Example 2.2. For $\lambda = 0$ and $\mu = 1$, a function $g \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{M}_\Sigma(0, 1, \alpha) \equiv \mathcal{N}_\Sigma(\alpha)$ if it satisfies the following conditions:

$$\Re\left(\frac{g(z)}{z}\right) > \alpha \quad \text{and} \quad \Re\left(\frac{h(w)}{w}\right) > \alpha,$$

where $0 \leqslant \alpha < 1$ and $z, w \in \Delta$ and the function h is given by (1.2).

Example 2.3. For $\lambda = 1$ and $\mu \geqslant 0$, a function $g \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{M}_\Sigma(1, \mu, \alpha) \equiv \mathcal{B}_\Sigma(\mu, \alpha)$ if it satisfies the following conditions:

$$\Re\left(g'(z)\left(\frac{g(z)}{z}\right)^{\mu-1}\right) > \alpha \quad \text{and} \quad \Re\left(h'(w)\left(\frac{h(w)}{w}\right)^{\mu-1}\right) > \alpha,$$

where $0 \leqslant \alpha < 1$ and $z, w \in \Delta$ and the function h is given by (1.2).

Example 2.4. For $\lambda = 1$ and $\mu = 0$, a function $g \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{M}_\Sigma(1, 0, \alpha) \equiv \mathcal{S}_\Sigma(\alpha)$ if it satisfies the following conditions:

$$\Re\left(\frac{zg'(z)}{g(z)}\right) > \alpha \quad \text{and} \quad \Re\left(\frac{wh'(w)}{h(w)}\right) > \alpha,$$

where $0 \leqslant \alpha < 1$ and $z, w \in \Delta$ and the function h is given by (1.2).

3. Coefficient estimates

For $g \in \Sigma$ given by (1.1), the inverse map $h = g^{-1}$ has the following Faber polynomial expansion:

$$h(w) = g^{-1}(w) = w + \sum_{n=0}^{\infty} \frac{B_n}{w^n} = w - b_0 - \sum_{n\geqslant 1} \frac{1}{n} K_{n+1}^n \frac{1}{w^n}; \quad w \in \Delta$$

where

$$K_{n+1}^n = nb_0^{n-1}b_1 + n(n-1)b_0^{n-2}b_2 + \frac{1}{2}n(n-1)(n-2)b_0^{n-3}\left(b_3 + b_1^2\right) + \frac{n(n-1)(n-2)(n-3)}{3!}b_0^{n-4}(b_4 + 3b_1b_2) + \sum_{j\geqslant 5} b_0^{n-j} V_j$$

and V_j with $5 \leqslant j \leqslant n$ is a homogeneous polynomial of degree j in the variables $b_1, b_2, \ldots, b_n$. (See [1,2] or [9,13].)

Consequently, for functions $g \in \mathcal{M}_\Sigma(\lambda, \mu, \alpha)$ of the form (1.1), we can write:

$$(1-\lambda)\left(\frac{g(z)}{z}\right)^{\mu} + \lambda g'(z)\left(\frac{g(z)}{z}\right)^{\mu-1} = 1 + \sum_{n=0}^{\infty} F_{n+1}(b_0, b_1, \ldots, b_n)\frac{1}{z^{n+1}}, \tag{3.1}$$

where

$$F_1 = (\mu - \lambda)b_0$$

$$F_2 = \frac{1}{2!}(\mu-1)(\mu-2\lambda)b_0^2 + (\mu-2\lambda)b_1$$

$$F_3 = \frac{1}{3!}(\mu-1)(\mu-2)(\mu-3\lambda)b_0^3 + (\mu-1)(\mu-3\lambda)b_0b_1 + (\mu-3\lambda)b_2$$

$$F_4 = \frac{1}{4!}(\mu-1)(\mu-2)(\mu-3)(\mu-4\lambda)b_0^4 + \frac{1}{2!}(\mu-1)(\mu-2)(\mu-4\lambda)b_0^2b_1 + (\mu-1)(\mu-4\lambda)\left[b_0b_2 + \frac{b_1^2}{2}\right] + (\mu-4\lambda)b_3$$

$$F_5 = \frac{1}{5!}(\mu-1)\cdots(\mu-4)(\mu-5\lambda)b_0^5 + \frac{1}{3!}(\mu-1)(\mu-2)(\mu-3)(\mu-5\lambda)b_0^3b_1 + \frac{1}{2!}(\mu-1)(\mu-2)(\mu-5\lambda)\left[b_0b_1^2 + b_0^2b_2\right] + (\mu-1)(\mu-5\lambda)[b_0b_3 + b_1b_2] + (\mu-5\lambda)b_4.$$

In general,

$$F_{n+1}(b_0, b_1, \ldots, b_n) = \left[\mu - (n+1)\lambda\right] \times \left[(\mu-1)!\right] \times \left[\sum_{i_1+2i_2+\cdots+(n+1)i_{n+1}=n+1} \frac{b_0^{i_1} b_1^{i_2} \cdots b_n^{i_{n+1}}}{i_1! i_2! \cdots i_{n+1}! [\mu - (i_1 + i_2 + \cdots + i_{n+1})]!}\right]$$

is a Faber polynomial of degree $n+1$.

Our first theorem introduces an upper bound for the coefficients $|b_n|$ of bi-univalent functions in $\mathcal{M}_\Sigma(\lambda, \mu, \alpha)$ having certain gap series.

Theorem 3.1. *For* $0 \leqslant \lambda \leqslant 1$, $\mu \geqslant 0$ *and* $0 \leqslant \alpha < 1$ *let* $g \in \mathcal{M}_\Sigma(\lambda, \mu, \alpha)$*. If* $b_k = 0$; $(0 \leqslant k \leqslant n-1)$ *then*

$$|b_n| \leqslant \left|\frac{2(1-\alpha)}{\mu - (n+1)\lambda}\right|.$$

Proof. For $g \in \mathcal{M}_\Sigma(\lambda, \mu, \alpha)$ we have the expansion (3.1) and for the inverse map $h = g^{-1}$ we obtain:

$$(1-\lambda)\left(\frac{h(w)}{w}\right)^{\mu} + \lambda h'(w)\left(\frac{h(w)}{w}\right)^{\mu-1} = 1 + \sum_{n=0}^{\infty} F_{n+1}(B_0, B_1, \ldots, B_n)\frac{1}{w^{n+1}}. \tag{3.2}$$

Imposing the asserted hypothesis $b_k = 0$; $(0 \leqslant k \leqslant n-1)$ in the theorem, (3.1) and (3.2) respectively imply:

$$(1-\lambda)\left(\frac{g(z)}{z}\right)^{\mu} + \lambda g'(z)\left(\frac{g(z)}{z}\right)^{\mu-1} = 1 + \left[\mu - (n+1)\lambda\right]\frac{b_n}{z^{n+1}}, \tag{3.3}$$

and

$$(1-\lambda)\left(\frac{h(w)}{w}\right)^{\mu} + \lambda h'(w)\left(\frac{h(w)}{w}\right)^{\mu-1} = 1 + \left[\mu - (n+1)\lambda\right]\frac{B_n}{w^{n+1}}. \tag{3.4}$$

On the other hand, for $g \in \mathcal{M}_\Sigma(\lambda, \mu, \alpha)$, by definition, there exist two positive real-part functions $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^{-n}$ and $q(w) = 1 + \sum_{n=1}^{\infty} q_n w^{-n}$ in Δ so that

$$(1-\lambda)\left(\frac{g(z)}{z}\right)^{\mu} + \lambda g'(z)\left(\frac{g(z)}{z}\right)^{\mu-1} = 1 + (1-\alpha)\sum_{n=0}^{\infty} K_{n+1}^{1}(p_1, p_2, \ldots, p_{n+1})\frac{1}{z^{n+1}} \tag{3.5}$$

and

$$(1-\lambda)\left(\frac{h(w)}{w}\right)^{\mu} + \lambda h'(w)\left(\frac{h(w)}{w}\right)^{\mu-1} = 1 + (1-\alpha)\sum_{n=0}^{\infty} K_{n+1}^{1}(q_1, q_2, \ldots, q_{n+1})\frac{1}{w^{n+1}}. \tag{3.6}$$

According to Caratheodory lemma (e.g., see [3]), we note that $|p_n| \leqslant 2$ and $|q_n| \leqslant 2$. Therefore, comparing the corresponding coefficients of (3.3) and (3.5), we get:

$$\left[\mu - (n+1)\lambda\right]b_n = (1-\alpha)K_{n+1}^{1}(p_1, p_2, \ldots, p_{n+1}). \tag{3.7}$$

Similarly, from (3.4) and (3.6), we obtain:

$$\left[\mu - (n+1)\lambda\right]B_n = (1-\alpha)K_{n+1}^{1}(q_1, q_2, \ldots, q_{n+1}). \tag{3.8}$$

Note that $B_n = -b_n$ if $b_k = 0$, $(0 \leqslant k \leqslant n-1)$ and so Eqs. (3.7) and (3.8) yield

$$\left[\mu - (n+1)\lambda\right]b_n = (1-\alpha)p_{n+1}, \tag{3.9}$$

$$-\left[\mu - (n+1)\lambda\right]b_n = (1-\alpha)q_{n+1}. \tag{3.10}$$

Solving either of Eqs. (3.9) or (3.10) for $|b_n|$ and applying the Caratheodory lemma, we obtain:

$$|b_n| \leqslant \frac{2(1-\alpha)}{|\mu - (n+1)\lambda|}.$$

The following corollaries are immediate consequences of the above theorem. □

Corollary 3.1. *Let $g(z)$ be given by (1.1). For $0 \leqslant \lambda \leqslant 1$ and $0 \leqslant \alpha < 1$ if $g \in \mathcal{F}_\Sigma(\lambda, \alpha)$ and $b_k = 0$, $(0 \leqslant k \leqslant n-1)$ then*

$$|b_n| \leqslant \frac{2(1-\alpha)}{|1-(n+1)\lambda|}, \quad (n \geqslant 1).$$

Corollary 3.2. *Let $g(z)$ be given by (1.1). For $0 \leqslant \alpha < 1$, if $g \in \mathcal{N}_\Sigma(\alpha)$ and $b_k = 0$, $(0 \leqslant k \leqslant n-1)$ then*

$$|b_n| \leqslant 2(1-\alpha), \quad (n \geqslant 1).$$

Corollary 3.3. *Let $g(z)$ be given by (1.1). For $\mu \geqslant 0$ and $0 \leqslant \alpha < 1$, if $g \in \mathcal{B}_\Sigma(\mu, \alpha)$ and $b_k = 0$, $(0 \leqslant k \leqslant n-1)$ then*

$$|b_n| \leqslant \frac{2(1-\alpha)}{|\mu-n-1|}, \quad (n \geqslant 1).$$

Corollary 3.4. *Let $g(z)$ be given by (1.1). For $0 \leqslant \alpha < 1$, if $g \in \mathcal{S}_\Sigma(\alpha)$ and $b_k = 0$, $(0 \leqslant k \leqslant n-1)$ then*

$$|b_n| \leqslant \frac{2(1-\alpha)}{|n+1|}, \quad (n \geqslant 1).$$

In the following theorem, we demonstrate the unpredictability of the coefficients of bi-univalent functions if the coefficient restriction imposed on Theorem 3.1 is relaxed.

Theorem 3.2. *Let g be given by (1.2). For $0 \leqslant \lambda \leqslant 1$, $\mu \geqslant 0$ and $0 \leqslant \alpha < 1$ if $g \in \mathcal{M}_\Sigma(\lambda, \mu, \alpha)$ then*

(i)

$$|b_0| \leqslant \begin{cases} \sqrt{\frac{4(1-\alpha)}{|(\mu-1)(\mu-2\lambda)|}}, & 0 \leqslant \alpha < 1 - \frac{(\mu-\lambda)^2}{|(\mu-1)(\mu-2\lambda)|}; \\ \frac{2(1-\alpha)}{|\mu-\lambda|}, & 1 - \frac{(\mu-\lambda)^2}{|(\mu-1)(\mu-2\lambda)|} \leqslant \alpha < 1. \end{cases}$$

(ii)

$$|b_1| \leqslant \frac{2(1-\alpha)}{|\mu-2\lambda|}.$$

Proof. Comparing the corresponding coefficients of Eqs. (3.5) and (3.6) we obtain:

$$(\mu-\lambda)b_0 = (1-\alpha)p_1 \tag{3.11}$$

and

$$-(\mu-\lambda)b_0 = (1-\alpha)q_1. \tag{3.12}$$

Taking the absolute values of either of (3.11) or (3.12) and applying the Caratheodory lemma, we obtain

$$|b_0| \leqslant \frac{2(1-\alpha)}{|\mu-\lambda|}.$$

On the other hand, comparing the corresponding coefficients of Eq. (3.5), for $n = 1$, we obtain:

$$\frac{1}{2}(\mu-1)(\mu-2\lambda)b_0^2 + (\mu-2\lambda)b_1 = (1-\alpha)p_2. \tag{3.13}$$

Similarly, from (3.6) we obtain:

$$\frac{1}{2}(\mu-1)(\mu-2\lambda)b_0^2 - (\mu-2\lambda)b_1 = (1-\alpha)q_2. \tag{3.14}$$

Adding Eqs. (3.13) and (3.14) yields:

$$(\mu-1)(\mu-2\lambda)b_0^2 = (1-\alpha)(p_2+q_2). \tag{3.15}$$

Solving (3.15) for $|b_0|$ and taking the absolute values of both sides, we obtain:

$$|b_0| = \sqrt{\frac{(1-\alpha)|p_2+q_2|}{|(\mu-1)(\mu-2\lambda)|}} \leqslant \sqrt{\frac{4(1-\alpha)}{|(\mu-1)(\mu-2\lambda)|}}.$$

Now part (i) of the theorem follows upon noting that

$$\sqrt{\frac{4(1-\alpha)}{|(\mu-1)(\mu-2\lambda)|}} < \frac{2(1-\alpha)}{|\mu-\lambda|}$$

if

$$0 \leqslant \alpha < 1 - \frac{(\mu-\lambda)^2}{|(\mu-1)(\mu-2\lambda)|}.$$

To prove the second part of the theorem, subtract (3.14) from (3.13) to obtain:

$$2(\mu-2\lambda)b_1 = (1-\alpha)(p_2-q_2). \tag{3.16}$$

Dividing (3.16) by $2(\mu-2\lambda)$ and taking the absolute values yield:

$$|b_1| = \frac{(1-\alpha)|p_2-q_2|}{2|\mu-2\lambda|} \leqslant \frac{2(1-\alpha)}{|\mu-2\lambda|}. \qquad \square$$

Corollary 3.5. *For* $0 \leqslant \alpha < 1$ *let* $g \in \mathcal{M}_\Sigma(\lambda,\mu,\alpha) \equiv \mathcal{M}_\Sigma(\alpha)$. *Then*

$$|b_0| \leqslant \begin{cases} \sqrt{2(1-\alpha)}, & 0 \leqslant \alpha < \frac{1}{2}; \\ 2(1-\alpha), & \frac{1}{2} \leqslant \alpha < 1. \end{cases}$$

We note that the bound given in Corollary 3.5 is an improvement to the bound given in [4, Theorem 2.(i)].

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