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Number theory

On small zeros of automorphic L-functions

Petits zéros des fonctions L de formes automorphes

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ABSTRACT

In this paper, we first formulate the Weil explicit formula of prime number theory for cuspidal automorphic L-functions $L(s,\pi)$ of GL_d. Then, we prove some conditional results about the vanishing order at the central point of $L(s,\pi)$. This enables to yield an estimate for the height of the lowest zero of $L(s,\pi)$ on the critical line in terms of the analytic conductor.

RÉSUMÉ

Dans cet article, nous formulons d'abord les formules explicites de Weil de la théorie des nombres premiers pour les fonctions L de formes automorphes cuspidales $L(s,\pi)$ de GL_d. Ensuite, nous montrons des résultats conditionnels concernant l'ordre d'annulation de $L(s,\pi)$ au point $s=1/2$, ce qui permet de donner une estimation de la hauteur du plus petit zéro de $L(s,\pi)$ sur la droite critique en termes de conducteur analytique.

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1. Introduction

Special values of L-functions often carry relevant arithmetic or geometric information on the objects that were used to define the L-functions. One is particularly interested in the vanishing or non-vanishing of various families of L-functions at $s=1/2$ in the standard normalization.

In this paper, we give under the Riemann hypothesis some estimates for the order n_π of an eventual zero of a cuspidal automorphic L-function $L(s,\pi)$ of GL_d at the point $s=1/2$ and for the height of the lowest zero of $L(s,\pi)$ on the critical line in terms of the analytic conductor. For this purpose, we shall first formulate Weil's explicit formula in the context of cuspidal automorphic L-functions. Let K be an algebraic number field of degree n, O_K the ring of integers and A_K the adele ring of K. Let S_f and S_∞ be the sets of all finite and infinite places of K, respectively. Write $S_\infty = S_\mathbb{R} \sqcup S_\mathbb{C}$, where $S_\mathbb{R}$ (resp. $S_\mathbb{C}$) is the set of all real (resp. complex) places of K and put $r_1 = \#S_\mathbb{R}$ (resp. $r_2 = \#S_\mathbb{C}$). Let $\pi = \bigotimes_v \pi_v$

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be an irreducible cuspidal automorphic representation of $GL_d(A_K)$. Then, from the general theory [3], we can define the L-function $L(s,\pi)$ by the Euler product:

$$L(s,\pi)=\prod_{v\in S_f}\prod_{j=1}^{d}\left(1-\alpha_{v,j}(\pi)q_v^{-s}\right)^{-1}=\sum_{n=1}^{\infty}\frac{\lambda_\pi(n)}{n^s}\quad\left(\mathrm{Re}(s)>1\right),$$

where q_v is the residue degree of the local field K_v with K_v being the v-adic completion of K at v and the complex number $\alpha_{v,j}(\pi)$ is determined by the local representation π_v for each $v\in S_f$. From the Euler product expression of $L(s,\pi)$, we get

$$-\frac{L'}{L}(s,\pi)=\sum_{v\in S_f}\sum_{l=1}^{\infty}\frac{\Lambda_\pi(q_v^l)}{q_v^{ls}},$$

where $\Lambda_\pi(q_v^l):=\log q_v\sum_{j=1}^{d}\alpha_{v,j}(\pi)^l$. Moreover, let $\Lambda(s,\pi)$ be the completed L-function defined by

$$\Lambda(s,\pi)=L_\infty(s,\pi)L(s,\pi),$$

where $L_\infty(s,\pi)$ is defined by

$$L_\infty(s,\pi)=\prod_{v\in S_\infty}\prod_{j=1}^{d}\Gamma_v\left(s+\mu_{v,j}(\pi)\right).$$

Here, $\Gamma_v(s)$ is defined by

$$\Gamma_v(s)=N_v(N_v\pi)^{-\frac{N_v s}{2}}\Gamma\left(\frac{N_v s}{2}\right)$$

with $N_v=1$ if $v\in S_{\mathbb{R}}$ and $N_v=2$ otherwise and $\mu_{v,j}(\pi)$ is a complex number determined by π_v for each $v\in S_\infty$. The number $d_\pi=d_{L(s,\pi)}=d\sum_{v\in S_\infty}N_v$ denotes the degree of the function $L(s,\pi)$. We note that $\mathrm{Re}(\mu_{v,j}(\pi))>-\frac{1}{2}$. It is known that $\Lambda(s,\pi)$ can be continued analytically to the whole plane $\mathbb{C}$ except in the case $d_\pi=1$, and that π is the trivial character **1** for which $L(s,\pi)$ is the Dedekind zeta function $\zeta_K(s)$ of K and $\Lambda(s,\pi)$ has simple poles at $s=0$ and $s=1$. Moreover, it satisfies the functional equation

$$N_\pi^{\frac{s}{2}}\Lambda(s,\pi)=\mathrm{e}_\pi N_\pi^{\frac{1-s}{2}}\Lambda(1-s,\overline{\pi}),$$

where $N_\pi\geq 1$ is called the conductor of π, e_π is the root number which is of modulus 1 and $\overline{\pi}$ is the contragradient representation of π. Since we look for uniform estimates for n_π and the height of the lowest zero of $L(s,\pi)$ on the critical line, it turns out that the results can be expressed conveniently in terms of the analytic conductor $\mathcal{N}_\pi$ [5, p. 713] defined by

$$\mathcal{N}_\pi=N_\pi\prod_{v\in S_\infty}\prod_{j=1}^{d}\left(1+\left|\mu_{v,j}(\pi)\right|^{N_v}\right).$$

The Generalized Ramanujan Conjecture (GRC) asserts that if v is a place where π_v is unramified, then $|\alpha_{v,j}(\pi)|=1$ and $\mathrm{Re}(\mu_{v,j}(\pi))=0$. Unconditionally, Jacquet and Shalika [6] proved the bounds

$$q_v^{-1/2}<\left|\alpha_{v,j}(\pi)\right|<q_v^{1/2},$$

and a similar local analysis for archimedean places would give $|\mathrm{Re}(\mu_{v,j}(\pi))|<\frac{1}{2}$. The best bound for general GL_d is due to Luo, Rudnick, and Sarnak [7]. The Ramanujan bound has been proven in very few cases. For instance, the most prominent among them are holomorphic forms on GL_2 and GSp_4. See [2] for a survey of what progress is known towards proving the Ramanujan bound.

2. The Weil explicit formula

The Weil explicit formula for an L-function is a tool that gives a relation between a function evaluated at the zeros of an L-function and the Fourier transform of that function evaluated at logarithms of prime powers, with some additional structure related to the global nature of the L-function. By following the strategy of Iwaniec and Kowalski [4, Section 5.5], we can formulate the following form of the explicit formula. For $T>0$, let $\mathcal{R}(\pi)$ be the set of non-trivial zeros of $L(s,\pi)$.

Lemma 1. *Let $Q>1$ and $\phi(x)$ be a function in the Schwartz space $\mathcal{S}(\mathbb{R})$ whose Fourier transform $\hat{\phi}(y)=\int_{-\infty}^{\infty}\phi(x)\mathrm{e}^{-2\pi\mathrm{i}xy}\,\mathrm{d}x$ has compact support (in particular, ϕ can be extended as a smooth function on $\mathbb{C}$). Then, it holds that*

$$\sum_{\rho\in\mathcal{R}(\pi)}\phi\left(\frac{\log Q}{2\pi \mathrm{i}}\left(\rho-\frac{1}{2}\right)\right)=\frac{\log N_\pi}{\log Q}\hat{\phi}(0)+\left[\phi\left(\frac{\log Q}{4\pi \mathrm{i}}\right)+\phi\left(-\frac{\log Q}{4\pi \mathrm{i}}\right)\right]\delta_{1,\mathbf{1}}+\frac{1}{\log Q}\sum_{v\in S_\infty}\sum_{j=1}^{d}H_{v,j}(Q,\phi,\pi)$$
$$-\frac{1}{\log Q}\sum_{v\in S_f}\sum_{l=1}^{\infty}\left(\frac{\Lambda_\pi(q_v^l)}{q_v^{\frac{l}{2}}}\hat{\phi}\left(\frac{l\log q_v}{\log Q}\right)+\frac{\Lambda_{\overline{\pi}}(q_v^l)}{q_v^{\frac{l}{2}}}\hat{\phi}\left(-\frac{l\log q_v}{\log Q}\right)\right),$$

where

$$H_{v,j}(Q,\phi,\pi)=\int_{-\infty}^{\infty}\phi(t)\left(\frac{\Gamma_v'}{\Gamma_v}\left(\frac{1}{2}+\mu_{v,j}(\pi)+\frac{2\pi \mathrm{i}t}{\log Q}\right)+\frac{\Gamma_v'}{\Gamma_v}\left(\frac{1}{2}+\mu_{v,j}(\overline{\pi})-\frac{2\pi \mathrm{i}t}{\log Q}\right)\right)\mathrm{d}t$$

and $\delta_{1,1}=\delta_{1,1}(\pi)=1$ *if* $d_\pi=1$ *or* $\pi=1$ *and 0 otherwise.*

Using the same argument as Barner [1], we deduce from Lemma 1 a similar form of the Weil-type explicit formula. For a function $F:\mathbb{R}\to\mathbb{C}$ of bounded variation (i.e., $V_{\mathbb{R}}(F)<\infty$ where $V_{\mathbb{R}}(F)$ is the total variation of F on $\mathbb{R}$), we define the function $\Phi_F(s)$ for $s\in\mathbb{C}$ by:

$$\Phi_F(s)=\hat{F}\left(-\frac{s-\frac{1}{2}}{2\pi \mathrm{i}}\right)=\int_{-\infty}^{\infty}F(x)\,\mathrm{e}^{(s-\frac{1}{2})x}\mathrm{d}x.$$

Moreover, for $v\in S_\infty$ and $1\le j\le d$, let $F_{v,j}(x,\pi)=F(x)\mathrm{e}^{-2\mathrm{i}\frac{\eta_{v,j}(\pi)}{N_v}x}$, $\widetilde{F}_{v,j}(x,\pi):=F_{v,j}(x,\pi)+F_{v,j}(-x,\pi)$ and $\mu_{v,j}(\pi)=\xi_{v,j}(\pi)+\mathrm{i}\eta_{v,j}(\pi)$ with $\xi_{v,j}(\pi),\eta_{v,j}(\pi)\in\mathbb{R}$.

Theorem 2.1. *Let $F:\mathbb{R}\to\mathbb{C}$ be a function of bounded variation that satisfies the following conditions:*

(a) *there is a positive constant b such that $V_{\mathbb{R}}(F(x)\mathrm{e}^{(\frac{1}{2}+b)|x|})<\infty$;*
(b) *F is "normalized", that is, $2F(x)=F(x+0)+F(x-0)$ for $x\in\mathbb{R}$;*
(c) *for any $v\in S_\infty$ and $1\le j\le d$, $\widetilde{F}_{v,j}(x,\pi)=2F(0)+O(|x|)$ as $|x|\to 0$.*

Then, we have

$$\sum_{\rho\in\mathcal{R}(\pi)}\Phi_F(\rho)=F(0)\log\frac{N_\pi}{(2^{2r_2}\pi^n)^d}+\big(\Phi_F(0)+\Phi_F(1)\big)\delta_{1,\mathbf{1}}+\sum_{v\in S_\infty}\sum_{j=1}^{d}W_{v,j}(F,\pi)$$
$$-\sum_{v\in S_f}\sum_{l=1}^{\infty}\left(\frac{\Lambda_\pi(q_v^l)}{q_v^{\frac{l}{2}}}F(l\log q_v)+\frac{\Lambda_{\overline{\pi}}(q_v^l)}{q_v^{\frac{l}{2}}}F(-l\log q_v)\right),\qquad(1)$$

where

$$W_{v,j}(F,\pi)=\int_{0}^{\infty}\left(\frac{N_vF(0)}{x}-\widetilde{F}_{v,j}(x,\pi)\frac{\mathrm{e}^{(\frac{2}{N_v}-\frac{1}{2}-\xi_{v,j}(\pi))x}}{1-\mathrm{e}^{-\frac{2}{N_v}x}}\right)\mathrm{e}^{-\frac{2}{N_v}x}\mathrm{d}x.$$

Proof. Replace $Q=\mathrm{e}^{2\pi}$ and $\phi(x)=\hat{F}(-\frac{x}{2\pi})$ in Lemma 1 and using that $\hat{\phi}(y)=2\pi F(2\pi y)$, we obtain

$$\sum_{\rho\in\mathcal{R}(\pi)}\Phi_F(\rho)=F(0)\log N_\pi+\big(\Phi_F(0)+\Phi_F(1)\big)\delta_{1,\mathbf{1}}+\sum_{v\in S_\infty}\sum_{j=1}^{d}Y_{v,j}(F,\pi)$$
$$-\sum_{v\in S_f}\sum_{l=1}^{\infty}\left(\frac{\Lambda_\pi(q_v^l)}{q_v^{\frac{l}{2}}}F(l\log q_v)+\frac{\Lambda_{\overline{\pi}}(q_v^l)}{q_v^{\frac{l}{2}}}F(-l\log q_v)\right),$$

where

$$Y_{v,j}(F,\pi)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{F}\left(-\frac{t}{2\pi}\right)\left(\frac{\Gamma_v'}{\Gamma_v}\left(\frac{1}{2}+\mu_{v,j}(\pi)+\mathrm{i}t\right)+\frac{\Gamma_v'}{\Gamma_v}\left(\frac{1}{2}+\mu_{v,j}(\overline{\pi})-\mathrm{i}t\right)\right)\mathrm{d}t.$$

Notice that both conditions (a) and (b) guarantee the convergence of the infinite sum $\sum_{\rho\in\mathcal{R}(\pi)}\Phi_F(\rho)$ (more precisely, see [1]). Now, we compute the integral $Y_{v,j}(F,\pi)$. Since $\mu_{v,j}(\overline{\pi})=\overline{\mu_{v,j}(\pi)}=\xi_{v,j}(\pi)-\mathrm{i}\eta_{v,j}(\pi)$ and using the formula $\frac{\Gamma_v'}{\Gamma_v}(s)=-\frac{N_v}{2}\log N_v\pi+\frac{N_v}{2}\frac{\Gamma'}{\Gamma}(\frac{N_v s}{2})$, we have:

$$\begin{aligned}
Y_{v,j}(F,\pi)&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\hat{F}\left(-\frac{t-\eta_{v,j}(\pi)}{2\pi}\right)+\hat{F}\left(\frac{t+\eta_{v,j}(\pi)}{2\pi}\right)\right]\frac{\Gamma_v'}{\Gamma_v}\left(\frac{1}{2}+\xi_{v,j}(\pi)+\mathrm{i}t\right)\mathrm{d}t\\
&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\widetilde{F}_{v,j}(\cdot,\pi)^{\wedge}\left(\frac{t}{2\pi}\right)\left(-\frac{N_v}{2}\log N_v\pi+\frac{N_v}{2}\frac{\Gamma'}{\Gamma}\left(\frac{N_v}{2}\left(\frac{1}{2}+\xi_{v,j}(\pi)+\mathrm{i}t\right)\right)\right)\mathrm{d}t\\
&=F(0)\log\frac{1}{(N_v\pi)^{N_v}}+\frac{N_v}{2}\frac{1}{2\pi}\int_{-\infty}^{\infty}\widetilde{F}_{v,j}(\cdot,\pi)^{\wedge}\left(\frac{t}{2\pi}\right)\frac{\Gamma'}{\Gamma}\left(\frac{N_v}{2}\left(\frac{1}{2}+\xi_{v,j}(\pi)\right)+\mathrm{i}\frac{N_v}{2}t\right)\mathrm{d}t. \qquad (2)
\end{aligned}$$

Here, for $a,b>0$ and $G\in L^1(\mathbb{R})$ satisfying $V_{\mathbb{R}}(G)<\infty$ and $G(x)=G(0)+O(|x|)$ as $s\to 0$, the following formula was also established in [1]:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{G}\left(\frac{t}{2\pi}\right)\frac{\Gamma'}{\Gamma}\left(a+\mathrm{i}\frac{t}{b}\right)\mathrm{d}t=\int_{0}^{\infty}\left(\frac{G(0)}{x}-\frac{b\mathrm{e}^{(1-a)bx}}{1-\mathrm{e}^{-bx}}G(-x)\right)\mathrm{e}^{-bx}\mathrm{d}x.$$

Using the assumption (c) of Theorem 2.1, we can apply the above formula with $G=\widetilde{F}_{v,j}$, $a=\frac{N_v}{2}(\frac{1}{2}+\xi_{v,j}(\pi))$ and $b=\frac{2}{N_v}$ and obtain:

$$Y_{v,j}(F,\pi)=F(0)\log\frac{1}{(N_v\pi)^{N_v}}+W_{v,j}(F,\pi).$$

This completes the proof. We may also point out that similar explicit formulas were established by Mestre [8] for rather general L-functions. □

3. The lowest zero of L-functions

Theorem 2.1 makes it possible to prove under the Riemann hypothesis that the lowest zero of $L(s,\pi)$ tends to 1/2 when the analytic conductor $\mathcal{N}_\pi$ is large. To do so, we first give a conditional improvement of the upper bound for the vanishing order n_π of $L(s,\pi)$ at $s=1/2$. This yields an estimate for the imaginary part γ_π of the lowest zero $\rho_\pi=1/2+\mathrm{i}\gamma_\pi$ of $L(s,\pi)$ distinct from $\frac{1}{2}$. For this purpose, we apply Theorem 2.1 to suitable functions with compact support. If we assume the Riemann hypothesis, then one can prove more precise estimates on γ_π. Such improvements have been also considered by Mestre [8] for the elliptic curve L-functions, the author [9] for Dedekind zeta functions and Iwaniec and Kowalski [4, Proposition 5.21] as an application of the positivity technique in the explicit formula.

Theorem 3.1. *Assuming the Riemann hypothesis, we have for large $\mathcal{N}_\pi$:*

$$n_\pi\ll\frac{\log\mathcal{N}_\pi}{\log(\frac{3}{d}\log\mathcal{N}_\pi)}\quad\text{and}\quad|\gamma_\pi|\ll\frac{1}{\log(\frac{3}{d}\log\mathcal{N}_\pi)}.$$

Proof. We first need an estimate for the sum over the finite places of K in (1). Let F be a function of support contained in $[-1,1]$ satisfying the hypotheses of Theorem 2.1 and let $F_T(x)=F(x/T)$, then $\widehat{F_T}(u)=T\widehat{F}(u)$. By using the classical prime number theorem one can prove the following estimate.

Lemma 2. *The sum over $v\in S_f$ in (1) is bounded as follows:*

$$\left|\sum_{v\in S_f}\sum_{l=1}^{\infty}\left(\frac{\Lambda_\pi(q_v^l)}{q_v^{\frac{l}{2}}}F_T(l\log q_v)+\frac{\Lambda_{\overline{\pi}}(q_v^l)}{q_v^{\frac{l}{2}}}F_T(-l\log q_v)\right)\right|\ll d\mathrm{e}^T.$$

Actually, since $|\alpha_{v,j}(\pi)|<q_v^{1/2}$, we have $|\Lambda_\pi(n)|\le d\Lambda(n)n^{\frac{1}{2}}$. Therefore, using the prime number theorem, the sum over $v\in S_f$ in (1) is bounded by

$$2d\sum_{\log n\le T}\Lambda(n)\ll d\mathrm{e}^T,$$

where the implied constant is absolute. Let f be a function defined by

$$f(x)=\begin{cases}1-|x| & \text{if } |x|<1\\ 0 & \text{otherwise.}\end{cases}$$

Then, f satisfies the hypothesis of Theorem 2.1 and

$$\widehat{f}(u)=\left(\frac{2\sin(u/2)}{u}\right)^2.$$

Therefore, by applying Theorem 2.1 to f_T, we obtain:

$$n_\pi T\le\delta_{1,\mathbf{1}}e^{T/2}-2\sum_{n\ge1}\frac{\operatorname{Re}(\Lambda_\pi(n))}{n^{\frac{1}{2}}}F_T(\log n)+O(\log\mathcal{N}_\pi). \tag{3}$$

By using Lemma 2 and replacing T by $\log(\frac{3}{d}\log\mathcal{N}_\pi)$ in (3), we have for large $\mathcal{N}_\pi$:

$$n_\pi\ll\frac{\log\mathcal{N}_\pi}{\log(\frac{3}{d}\log\mathcal{N}_\pi)}.$$

Then, the first assertion of Theorem 3.1 is proved. In order to prove the second assertion of Theorem 3.1, we need another even function supported on $[-1,1]$. Let g be an even function defined for $x\ge0$ by

$$g(x)=\begin{cases}(1-x)\cos\pi x+\frac{3}{\pi}\sin\pi x & \text{if } x\in[0,1]\\ 0 & \text{otherwise.}\end{cases}$$

Note that g satisfies the conditions of Theorem 2.1, then

$$\widehat{g}(u)=\left(2-\frac{u^2}{\pi^2}\right)\left[\frac{2\pi}{\pi^2-u^2}\cos\frac{u}{2}\right]^2.$$

Applying Theorem 2.1 with $g_T(x)=g(x/T)$ and replacing T by $\sqrt{2}\pi/|\gamma_\pi|$, we obtain:

$$\frac{8}{\pi^2}n_\pi T-\big(\Phi_{g_T}(0)+\Phi_{g_T}(1)\big)\delta_{1,\mathbf{1}}+2\sum_{n\ge1}\frac{\operatorname{Re}(\Lambda_\pi(n))}{n^{\frac{1}{2}}}g_T(\log n)\gg\log\mathcal{N}_\pi. \tag{4}$$

Using Lemma 2, the last estimate of n_π, we deduce from (4) the following inequality for some constants A and B:

$$\frac{\log\mathcal{N}_\pi}{\log(\frac{3}{d}\log\mathcal{N}_\pi)}AT+Bde^T\gg\log\mathcal{N}_\pi.$$

Therefore, for sufficiently large $\mathcal{N}_\pi$, we get

$$T\gg\log\left(\frac{3}{d}\log\mathcal{N}_\pi\right),$$

so

$$|\gamma_\pi|\ll\frac{1}{\log(\frac{3}{d}\log\mathcal{N}_\pi)}.$$

As a consequence, one can show that any fixed interval on the critical line around $s=\frac{1}{2}$ contains zeros of $L(s,\pi)$ when $\mathcal{N}_\pi$ is sufficiently large. □

Corollary 1. *Assuming the Riemann hypothesis, we have:*

$$\lim_{\mathcal{N}_\pi\to+\infty}\rho_\pi=\frac{1}{2}.$$

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