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An explicit semi-factorial compactification of the Néron model

Une compactification semi-factorielle explicite du modèle de Néron

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ABSTRACT

C. Pépin recently constructed a semi-factorial compactification of the Néron model of an Abelian variety using the flattening technique of Raynaud–Gruson. Here we prove that an explicit semi-factorial compactification is a certain moduli space of sheaves — the family of compactified Jacobians.

RÉSUMÉ

C. Pépin a construit récemment une compactification semi-factorielle du modèle de Néron d'une variété abélienne en utilisant les techniques de platification de Raynaud–Gruson. Nous montrons ici qu'une compactification semi-factorielle explicite constitue un certain espace de modules de faisceaux – la famille de jacobiens compacifiés.

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We prove that the family of compactified Jacobians is a semi-factorial compactification of the Néron model of the Jacobian. Semi-factoriality is a weakening of factoriality, the condition that the local rings are unique factorization domains. In [18], Pépin introduced the condition and proved that the Néron model of an Abelian variety A_K over the field of fractions K of a discrete valuation ring R admits a semi-factorial compactification.

Pépin constructed the compactification using the flattening technique of Raynaud–Gruson [19]. We give an alternative construction when $A_K = J_K$ is a Jacobian satisfying suitable hypotheses. We prove that an explicit semi-factorial compactification is given by a compactification of J_K as a moduli space — by the family of compactified Jacobians.

What is the compactified Jacobian? Suppose $A_K = J_K$ is the Jacobian of the smooth curve X_K. The curve X_K extends to a regular model X/S over $S = \text{Spec}(R)$. The Jacobian J_K is the moduli space of degree 0 line bundles on X_K, and we can try to extend it to a family $\bar{J}/S$ by adding over the point $0 \in S$ a moduli space of sheaves on X_0. When X_0 is geometrically integral, we can extend J_K by adding the moduli space of degree 0 rank 1, torsion-free sheaves on X_0, and this extended family is the family of compactified Jacobians.

The line bundle locus J/S in a family of compactified Jacobians $\bar{J}/S$ is canonically isomorphic to the Néron model of J_K by (a special case of) [12, Theorem 3.9], a result that extends earlier work on the topic [17,3–6,15]. Compactified Jacobians

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are proper by construction, so $\overline{J}/S$ is a compactification of the Néron model. When the Picard rank of J_K is 1, $\overline{J}/S$ has the desirable properties studied by Pépin:

Main Theorem. *The Altman–D'Souza–Kleiman family of compactified Jacobians $\overline{J}/S$ is a semi-factorial model of the Néron model provided the Picard rank of J_K is* 1*.*

The Main Theorem includes the explicit hypothesis that J_K has Picard rank 1 and the implicit hypothesis that the special fiber X_0 is geometrically integral. How are these hypotheses used? When the hypothesis that X_0 is geometrically integral fails, the Altman–D'Souza–Kleiman family $\overline{J}/S$ is not defined because the moduli space of degree 0 rank 1, torsion-free sheaves on X_0 is badly behaved. A well-behaved space can be recovered by imposing e.g. a stability condition, but the proof we give here does not immediately apply to these more general spaces. In proving the Main Theorem, we use the property that translation $\tau_{a_K}: J_K \to J_K$ by a point $a_K \in J_K(K)$ extends to an automorphism $\overline{J} \to \overline{J}$. It is not known if $\overline{J}$ has this extension property when X_0 is reducible; the issue is that, when X_0 is reducible, the tensor product of two slope semi-stable line bundles can fail to be semi-stable.

The hypothesis that J_K has Picard rank 1 is used to assert that the Néron–Severi group $\mathrm{NS}(J_{\overline{K}})$ is generated by classes that extend to $\overline{J}$. Under the rank 1 hypothesis, $\mathrm{NS}(J_{\overline{K}})$ is generated by the class of the theta divisor, and Esteves and Soucaris have (independently) shown that this divisor extends. In general, when $\mathrm{NS}(J_{\overline{K}})$ is generated by classes that extend, our proof shows that $\overline{J}$ is semi-factorial, and it would be desirable to have more general results describing when classes in $\mathrm{NS}(J_{\overline{K}})$ extend.

1. Preliminaries

Here we collect results from the literature. Fix a discrete valuation ring (or dvr for short) R with field of fractions K and residue field $k(0)$. Set $S = \mathrm{Spec}(R)$ and $0 = \mathrm{Spec}(k(0))$. We fix a smooth *curve* $X_K/\mathrm{Spec}(K)$ (i.e. a K-scheme of pure dimension 1 that is proper, smooth, and geometrically connected over K) that we assume has genus $g \geq 1$ and study the associated *Jacobian* $J_K/\mathrm{Spec}(K)$. The Jacobian is a g-dimensional Abelian variety that represents the étale sheaf parameterizing degree 0 line bundles on X_K, and it extends to the Néron model J/S, a certain (possibly nonproper) S-scheme. We omit the definition, but one consequence, which we will use, is that the restriction map $J(S) \to J_K(K)$ is surjective, i.e. the weak Néron Mapping Property holds.

To study compactifications of J_K, we make the following definitions.

Definition 1. An S-scheme V/S is *semi-factorial* if the restriction map

$$\mathrm{Pic}(V) \to \mathrm{Pic}(V_K) \tag{1}$$

on Picard groups is surjective.

If V/S is separated and of finite type over S, then an *S-compactification* of V/S is a proper S-scheme $\overline{V}/S$ and an S-immersion $V \to \overline{V}$ with dense image. An S-compactification is a *semi-factorial model* if $\overline{V}/S$ is flat and projective over S, normal, and semi-factorial. A semi-factorial model is a *regular model* if $\overline{V}$ is a regular scheme.

Corollaire 6.4 of [18] states that the Néron model J/S admits a semi-factorial model. In fact, the Corollaire states that the semi-factorial model can be chosen to have certain desirable base-change properties, which we discuss in Remark 5.

The curve X_K admits a regular model X/S because resolution of singularities holds for arithmetic surfaces [14]. (Lipman's result is stated for R excellent, but the argument on [7, page 87] shows that this hypothesis can be removed.) For the remainder of this paper, we fix a regular model X/S satisfying

Assumption. X/S is a regular model of X_K with geometrically integral special fiber.

With this assumption, the Altman–D'Souza–Kleiman *family of compactified Jacobians* $\overline{J}/S$ associated to X/S is defined. The family of compactified Jacobians is an S-scheme $\overline{J}/S$ that is projective over S and represents the étale sheaf parameterizing families of degree 0 rank 1, torsion-free sheaves on X/S [1, (8.10) Theorem]. (Under more restrictive hypotheses, this is [8, Theorem II.4.1].) The line bundle locus in $\overline{J}$ is an open subscheme J that is the Néron model of J_K [12, Theorem 3.9].

We now recall the definition of the Néron–Severi group and the Picard scheme of J_K. The *Picard scheme* $\mathrm{Pic}(J_K/K)/\mathrm{Spec}(K)$ is a K-group scheme that is locally of finite type over K and represents the étale sheaf parameterizing line bundles on J_K. The line bundles that are algebraically equivalent to zero are parameterized by the identity component $\mathrm{Pic}^0(J_K/K)$ of the Picard scheme, which is an open and closed K-subgroup scheme that is of finite type over K.

Algebraic equivalence classes of line bundles on J_K form the *Néron–Severi group*, which is defined as

$$\mathrm{NS}(J_{\overline{K}}) := \frac{\mathrm{Pic}(J_K/K)(\overline{K})}{\mathrm{Pic}^0(J_K/K)(\overline{K})}$$

for $\overline{K}$ a fixed algebraic closure of K. This group is finitely generated, hence has a well-defined rank called the *Picard rank*.

The Picard rank of J_K is at least 1. Indeed, $J_{\overline{K}}$ admits a special type of divisor: the classical theta divisor. If $\mathcal{N}_{\overline{K}}$ is a line bundle of degree $g-1$ on $X_{\overline{K}}$, then

$$\Theta_{\overline{K}} := \{[\mathcal{L}_{\overline{K}}]:\ h^0(X_{\overline{K}}, \mathcal{L}_{\overline{K}} \otimes \mathcal{N}_{\overline{K}}) \neq 0\} \subset J_{\overline{K}}$$

is an ample divisor that defines a principal polarization. That is, the homomorphism

$$\phi: J_{\overline{K}} \to \operatorname{Pic}^0(J_{\overline{K}}/\overline{K}) \quad \text{defined by}$$
$$\phi(a) = \mathcal{O}_{J_{\overline{K}}}\big(\tau_a^*(\Theta_{\overline{K}}) - \Theta_{\overline{K}}\big) \tag{2}$$

is an isomorphism. Here τ_a is the translation-by-a map.

The divisor $\Theta_{\overline{K}}$ depends on the choice of $\mathcal{N}_{\overline{K}}$, but its image in the Néron–Severi group is independent of the choice, and we denote this common image by θ. Because $\Theta_{\overline{K}}$ is a principal polarization, θ is nonzero, and furthermore:

Lemma 2. *The class θ freely generates* $\mathrm{NS}(J_{\overline{K}})$ *when the Picard rank of J_K is* 1.

Proof. If J_K has Picard rank 1, then the Néron–Severi group $\mathrm{NS}(J_{\overline{K}})$ is cyclic because it is torsion-free [16, Corollary 2, page 178], so we may fix a generator c. Writing $\theta = n \cdot c$ for some $n \in \mathbf{Z}$, we have

$$\begin{aligned} n^g \cdot \big(c^g/g!\big) &= \theta^g/g! \\ &= 1 \quad \text{by the Riemann–Roch Formula.} \end{aligned}$$

So n^g divides 1 and hence $n = \pm 1$. □

2. Proof of the Main Theorem

Here we prove that $\overline{J}/S$ is a semi-factorial model of the Néron model provided the Picard rank of J_K is 1.

Lemma 3. *$\overline{J} \to S$ is flat, and $\overline{J}$ is Cohen–Macaulay and normal.*

Proof. Theorem (9) of [2] states that $\overline{J} \to S$ is flat with Cohen–Macaulay fibers. (That theorem includes the hypothesis that X lies on an S-smooth family of surfaces, but we can reduce to this case by arguing as in the proof of [10, Lemma 3.4].) Since S is Cohen–Macaulay, we can conclude that $\overline{J}$ itself is Cohen–Macaulay.

We prove $\overline{J}$ is normal using Serre's criteria. To verify the criteria, we need to show that Condition R1 holds. The line bundle locus $J_0 \subset \overline{J}_0$ is dense in the special fiber by [2, Theorem (9)], so the line bundle locus $J \subset \overline{J}$ in the total space contains all codimension 1 points. The locus J is contained in the smooth locus of $\overline{J}/S$, hence in the regular locus of $\overline{J}$, and so Condition R1 is satisfied. □

Proof of the Main Theorem. By Lemma 3 we just need to show that $\overline{J}/S$ is semi-factorial, i.e.

$$\operatorname{Pic}(\overline{J}) \to \operatorname{Pic}(J_K) \tag{3}$$

is surjective.

First, assume that X admits a line bundle $\mathcal{N}$ with fiber-wise degree $g-1$. Then the set $\{[\mathcal{L}] \in \overline{J}:\ h^0(X, \mathcal{L} \otimes \mathcal{N}) \neq 0\} \subset \overline{J}$ is the support of a relatively effective divisor Θ that extends the classical theta divisor by [20, Theorem 13] (or [9, page 184]). In particular, $\mathcal{O}_{J_K}(\Theta_K)$ lies in the image of (3).

That image also contains all line bundles algebraically equivalent to zero. Indeed, the polarization isomorphism ϕ from Eq. (2) is defined over K, so if $\mathcal{L}_K$ is a line bundle on J_K that is algebraically equivalent to zero, then we can write $[\mathcal{L}_K] = \phi(a_K)$ for some $a_K \in J_K(K)$. Here $[\mathcal{L}_K] \in \operatorname{Pic}^0(J_K/K)(K)$ is the point represented by $\mathcal{L}_K$. The S-scheme J/S satisfies the Néron Mapping Property (by e.g. [12, Theorem 3.9]), so $a_K \in J_K(K)$ is the restriction of some $a \in J(S)$. The line bundle locus J acts on $\overline{J}$ (by tensor product), so translation $\tau_a: \overline{J} \to \overline{J}$ by a is well-defined, and the line bundle $\mathcal{L} := \mathcal{O}_{\overline{J}}(\tau_a^*(\Theta) - \Theta)$ extends $\mathcal{L}_K$.

We have now shown that the image of (3) contains both $\mathcal{O}_{J_K}(\Theta_K)$ and the line bundles algebraically equivalent to zero. Together these line bundles generate $\operatorname{Pic}(J_K)$ by Lemma 2, so (3) is surjective, proving the theorem in the special case that an $\mathcal{N}$ exists.

In the general case, we argue as allows. Given a line bundle $\mathcal{L}_K$ on J_K, we can extend $\mathcal{L}_K$ to a family $\mathcal{L}$ of rank 1, torsion-free sheaves on $\overline{J}$ (by e.g. the S-projectivity of the relevant compactified Picard scheme). There exists a line bundle $\mathcal{N}$ with fiber-wise degree $g-1$ on X_T for some étale cover $T \to S$ with T the spectrum of a dvr because X_0 is geometrically reduced. Say L is the field of fractions of the dvr $\Gamma(T, \mathcal{O}_T)$. The base-change X_T remains regular, so $\mathcal{L}_L$ extends to a line bundle on $\overline{J}_T$. This extension must equal $\mathcal{L}_T$ (by e.g. the S-separateness of the relevant compactified Picard scheme), so $\mathcal{L}_T$ and hence $\mathcal{L}$ must be a line bundle. □

Remark 4. Does $\overline{J}$ satisfy stronger conditions than semi-factoriality? Typically $\overline{J}$ does not satisfy the condition of regularity. Let $K = \mathbf{Q}$, $R = \mathbf{Z}_{(3)}$ (the localization of $\mathbf{Z}$ at 3), $S = \operatorname{Spec}(R)$, and X/S the minimal proper regular model of the affine curve $\operatorname{Spec}(R[x,y]/(y^2 - x^2(x-1)^2(x^2+1) - 3))$. The family X/S is a family of genus 2 curves with special fiber X_0 a rational curve with 2 nodes. Consider the family of compactified Jacobians $\overline{J}/S$ associated to X/S.

If $\nu: \mathbf{P}^1 \cong \widetilde{X}_0 \to X_0$ is the normalization, then $\overline{J}$ has a singularity at the rank 1, torsion-free sheaf $I := \nu_* \mathcal{O}(-2)$. The singularity of $\overline{J}$ at I is computed in [13]. The sheaf I fails to be locally free at 2 nodes, so by [13, Lemma 6.2] the completed local ring is isomorphic to

$$\widehat{O}_{\overline{J},[I]} = \widehat{R}\big[[a_1, b_1, a_2, b_2]\big]/(a_1 b_1 - 3, a_2 b_2 - 3).$$

This ring not only fails to be regular, but it also fails to be factorial. (The height 1 prime (a_1, a_2) is nonprincipal because the images of a_1, a_2 in the quotient $(3, a_i, b_i)/(3, a_i, b_i)^2$ are linearly independent.)

However, $\overline{J}/S$ is semi-factorial. Indeed, by the Main Theorem, we just need to show that $J_K = J_{\mathbf{Q}}$ has Picard rank 1, and we do so as follows. The Néron–Severi group $\mathrm{NS}(J_{\bar{\mathbf{Q}}})$ injects into the endomorphism ring $\operatorname{End}(J_{\bar{\mathbf{Q}}})$, and we compute this endomorphism ring by relating it to the endomorphism ring of the reduction of $J_{\mathbf{Q}}$ at a prime of good reduction.

Both the curve $X_{\mathbf{Q}}$ and its Jacobian $J_{\mathbf{Q}}$ have good reduction at the primes $p = 5, 13$, as can be seen by reducing the equation $y^2 = x^2(x-1)^2(x^2+1) + 3 \bmod p$. Using this equation to naively count $\mathbf{F}_{p^n}$-points, we compute that the characteristic polynomial f_p of the Frobenius endomorphism of $J_{\mathbf{F}_p}$ is

$$f_5 = x^4 - 2x^3 + 3x^2 - 10x + 25,$$
$$f_{13} = x^4 + 7x^3 + 35x^2 + 91x + 169.$$

Applying [11, Theorem 6] to these polynomials, we get that the reduction $J_{\mathbf{F}_p}$ is absolutely simple for $p = 5, 13$, so $\mathbf{Q} \otimes \operatorname{End}(J_{\bar{\mathbf{F}}_p}) = \mathbf{Q}[x]/(f_p)$.

The reduction map injects $\mathbf{Q} \otimes \operatorname{End}(J_{\bar{\mathbf{Q}}})$ into $\mathbf{Q} \otimes \operatorname{End}(J_{\bar{\mathbf{F}}_p})$ for $p = 5, 13$. A computation shows that the discriminant of $\mathbf{Q}[x]/(f_5)$ is coprime to the discriminant of $\mathbf{Q}[x]/(f_{13})$, and $\mathbf{Q}$ has no nontrivial unramified extensions, so the only field contained in both $\mathbf{Q}[x]/(f_5)$ and $\mathbf{Q}[x]/(f_{13})$ is $\mathbf{Q}$. In particular, $\operatorname{End}(J_{\bar{\mathbf{Q}}}) = \mathbf{Z}$. This example was suggested to the author by Bjorn Poonen.

Remark 5. Corollaire 6.4 of [18] proves that a semi-factorial model $\widetilde{J}/S$ of J_K can be chosen to be well-behaved with respect to certain dvr extensions. To be precise, given morphisms $T_1 \to S, \ldots, T_n \to S$ corresponding to extensions of R contained in the strict henselization R^{hs}, a semi-factorial model $\widetilde{J}/S$ can be chosen so that $\widetilde{J}_T/T$ is a semi-factorial model when $T \to S$ equals either some $T_i \to S$ or a morphism corresponding to a "*permise*" dvr extension.

The family $\overline{J}/S$ of compactified Jacobians satisfies this condition. In fact, it satisfies a stronger condition. By definition the formation of the family of compactified Jacobians commutes with arbitrary base change, so if $T \to S$ is a morphism that corresponds to a dvr extension, then $\overline{J}_T/T$ is a semi-factorial model of the Néron model provided X_T is regular. The scheme X_T is regular when $T \to S$ is one of the morphisms considered by Pépin or more generally when $T \to S$ is regular and surjective (see [18, Remarque 5.5]).

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