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Differential geometry

Harmonic vector fields on Landsberg manifolds

Champs de vecteurs harmoniques sur les variétés landsbergiennes

Alireza Shahi [1], Behroz Bidabad [*]

Faculty of Mathematics, Amirkabir University of Technology (Tehran Polytechnic), 424 Hafez Avenue, 15914 Tehran, Iran

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ABSTRACT

Let (M, F) be a compact boundaryless Landsberg manifold. In this work, a necessary and sufficient condition for a vector field on (M, F) to be harmonic is obtained. Next, on a compact boundaryless Finsler manifold of zero flag curvature, a necessary and sufficient condition for a vector field to be harmonic is found. Furthermore, the nonexistence of harmonic vector fields on a compact Landsberg manifold is studied and an upper bound for the first de Rham cohomology group is obtained.

RÉSUMÉ

Soit (M, F) une variété landsbergienne compacte sans bord. Dans cet article, il est obtenu une condition nécessaire et suffisante pour qu'un champ de vecteurs sur (M, F) soit harmonique. On donne ensuite un énoncé analogue sur une variété finslérienne compacte sans bord. En outre, on étudie la non-existence de champs de vecteurs harmoniques sur les variétés landsbergiennes compactes et, enfin, une borne supérieure pour le premier groupe de cohomologie de de Rham est obtenue.

1. Introduction

On a 2-dimensional Riemannian manifold, a harmonic vector field is a vector field for which *divergence* and *curl* operators vanish. In a compact Riemannian manifold, the existence of harmonic vector fields is closely related to the sign of the Ricci curvature and the topology of the underlying manifold. Bochner in [6] used the Laplace–Beltrami operator to prove some theorems on nonexistence of harmonic vector fields on compact Riemannian manifolds with positive or negative Ricci curvature. For instance, he proved that on a compact Riemannian manifold without boundary and with positive Ricci curvature, there is no vector field for which *divergence* and *differential* operators vanish simultaneously. Yano in [8] obtained a general formula and proved some of Bochner's theorems in compact Riemannian manifolds. Next, Wu in [7] proved that in Bochner's theorems, the assumption of positivity or negativity of the Ricci curvature can be replaced by quasi-positivity or

* Corresponding author.
E-mail addresses: alirezashahi@aut.ac.ir (A. Shahi), bidabad@aut.ac.ir (B. Bidabad).
URL: http://aut.ac.ir/bidabad (B. Bidabad).
[1] Tel./fax: +98 21 66 49 79 30.

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quasi-negativity, respectively. Akbar-Zadeh in [1] generalized Bochner and Yano's techniques in order to study certain vector fields on a compact Finsler manifold without boundary. Bao and Lackey in [3] construct a Laplace operator on differential forms and study harmonic forms on the underlying Finsler manifold. Next Zhong, C. and Zhong T., in [9] obtained an explicit formula for a horizontal Laplace operator.

In the present work we generalize some results of Yano for Landsberg manifolds. More intuitively a necessary and sufficient condition for a vector field to be harmonic is obtained, and the nonexistence of harmonic vector fields under certain conditions on a Landsberg manifold is proved. Moreover, on an n-dimensional compact Landsberg manifold, the dimension of the first de Rham cohomology group is at most $2n-1$ under a certain condition.

2. Preliminaries

Let (M,F) be a Finsler manifold, $\pi : TM_0 \to M$ the bundle of non-zero tangent vectors and π^*TM the pullback bundle. We adopt here more often the notations and the terminology of [2] and sometimes those of [4].

Denote the covariant derivatives of Cartan and Berwald connections by ∇ and D respectively. It is well known that the Whitney sum $TTM_0 = HTM \oplus VTM$, where HTM and VTM are horizontal and vertical bundles, respectively, and for any $\hat{X} \in TTM_0$ we have $\hat{X} = H\hat{X} + V\hat{X}$. Let X and Y be two sections on π^*TM. The relations between Cartan and Berwald connections are given by

$$D_{H\hat{X}}Y = \nabla_{H\hat{X}}Y + y^i(\nabla_i T)(X,Y), \tag{1}$$

$$D_{V\hat{X}}Y = V\hat{X}.Y, \tag{2}$$

where T is the Cartan tensor with the components $T_{kij} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}$ and $y = y^i \frac{\partial}{\partial x^i} \in T_xM$. By means of (1) we have $D_k g_{ij} = -2\nabla_0 T_{kij}$, where index 0 denotes the contracted multiplication by y, hence $D_0 g_{ij} = y^k D_k g_{ij} = -2y^k\nabla_0 T_{kij} = 0$. Eq. (2) is written locally $D_{\dot{\partial}_j} Y^i = \frac{\partial Y^i}{\partial y^j}$. The Ricci identities for Berwald connection are

$$D_l D_k X^i - D_k D_l X^i = X^r R^i_{r\ lk} - \frac{\partial X^i}{\partial y^r} R^r_{0\ lk}, \tag{3}$$

$$D_i D_k X_j - D_k D_i X_j = -X_l R^l_{j\ ik} - \frac{\partial X_j}{\partial y^r} R^r_{0\ ik}. \tag{4}$$

The Finsler manifold (M,F) is called *Landsberg* manifold if the hv-curvature P vanishes everywhere or equivalently $\nabla_0 T = 0$. Let $X = X^i(x)\frac{\partial}{\partial x^i}$ be a vector field on M. One can associate with X the 1-form $\tilde{X}$ on SM defined by $\tilde{X} = X_i(z)dx^i + \dot{X}_i dy^i$, where $\dot{X}_i = \frac{D_0 X_i - y_i D_0(y^j X_j)F^{-2}}{F}$. The horizontal part of its associated 1-form on SM is denoted again in this paper by $X = X_i(z)dx^i$, where $z \in SM$, cf., [2], p. 231, and the restriction of the first de Rham cohomology group to these 1-forms by $H^1_{dR}(SM)$. The differential and co-differential operators of the horizontal 1-form X are given by

$$dX = \frac{1}{2}(D_i X_j - D_j X_i)dx^i \wedge dx^j - \frac{\partial X_i}{\partial y^j}dx^i \wedge dy^j, \tag{5}$$

$$\delta X = -\left(\nabla^j X_j - X_j \nabla_0 T^j\right) = -g^{ij}D_i X_j, \tag{6}$$

where the co-differential operator δ is the formal adjoint of d, in the global scalar product over SM, cf., [2] pp. 223 and 239. Let (M,F) be a compact Finsler manifold without boundary, the divergence formula for a horizontal 1-form $X = X_i(z)dx^i$ is given by

$$\int_{SM} (\delta X)\eta = -\int_{SM} \left(g^{ij}D_i X_j\right)\eta = 0, \tag{7}$$

where η is a volume form on SM defined by

$$\eta(g) = \frac{(-1)^{\frac{n(n-1)}{2}}}{(n-1)!}\omega \wedge \overbrace{d\omega \wedge \ldots \wedge d\omega}^{(n-1)\text{-}time},$$

$\omega = u_i dx^i$ is the 1-form corresponding to a unitary vector field $u : M \to SM$.

3. Harmonic vector fields

Let (M,F) be a Finsler manifold, the vector field X on M is said to be *harmonic* if its corresponding horizontal 1-form on SM satisfies $\Delta X = d\delta(X) + \delta d(X) = 0$ or $dX = 0$ and $\delta X = 0$, equivalently:

$$D_i X_j = D_j X_i, \qquad g^{ij}D_i X_j = 0, \qquad \frac{\partial X_i}{\partial y^j} = 0. \tag{8}$$

Theorem 3.1. *If the vector field X on a compact Landsberg manifold without boundary satisfies $g^{jk}D_kD_jX^i=(R^i_k+T^{it}_rR^r_{0\ kt})X^k$ and $\frac{\partial X_i}{\partial y^j}=0$, then X is harmonic. The converse is true if $T^{it}_rR^r_{0\ kt}X^k=0$.*

Proof. Let $g^{jk}D_kD_jX^i=(R^i_k+T^{it}_rR^r_{0\ kt})X^k$ and $\frac{\partial X_h}{\partial y^j}=0$. By covariant derivative, we have:

$$D_j\big(X^kD_kX^j\big)=X^kD_jD_kX^j+D_kX^jD_jX^k. \tag{9}$$

Note that

$$\begin{aligned}\frac{\partial X^i}{\partial y^r}&=D_{\dot\partial_r}X^i=D_{\dot\partial_r}\big(X_hg^{ih}\big)\\&=X_hD_{\dot\partial_r}g^{ih}+g^{ih}D_{\dot\partial_r}X_h=X_hT^{ih}_r+g^{ih}\frac{\partial X_h}{\partial y^r}.\end{aligned} \tag{10}$$

By means of the torsion freeness of the Berwald connection, the assumption $\frac{\partial X_h}{\partial y^j}=0$, (3) and (10) we obtain:

$$D_lD_kX^i-D_kD_lX^i=R^i_{j\ lk}X^j-X_hT^{ih}_rR^r_{0\ lk}.$$

By contraction of the above equation with respect to i and l, we obtain:

$$D_jD_kX^j-D_kD_jX^j=R_{jk}X^j+X^jT^t_{rj}R^r_{0\ kt},$$

or equivalently

$$D_jD_kX^j=D_kD_jX^j+\big(R_{jk}+T^t_{rj}R^r_{0\ kt}\big)X^j. \tag{11}$$

Replacing (11) in (9), we find:

$$D_j\big(X^kD_kX^j\big)=X^kD_kD_jX^j+\big(R_{jk}+T^t_{rj}R^r_{0\ kt}\big)X^jX^k+D_kX^jD_jX^k. \tag{12}$$

The covariant derivative of $X^kD_jX^j$ yields:

$$D_k\big(X^kD_jX^j\big)=X^kD_kD_jX^j+D_kX^kD_jX^j. \tag{13}$$

Using (12) and (13) we have:

$$D_j\big(X^kD_kX^j\big)-D_k\big(X^kD_jX^j\big)=\big(R_{jk}+T^t_{rj}R^r_{0\ kt}\big)X^jX^k+D_kX^jD_jX^k-D_kX^kD_jX^j. \tag{14}$$

Since M is compact and the two terms in the left-hand side of (14) are divergence, by integration of (14) over SM and using divergence formula (7), we obtain:

$$\int_{SM}\big[\big(R_{jk}+T^t_{rj}R^r_{0\ kt}\big)X^jX^k+D_kX^jD_jX^k-D_kX^kD_jX^j\big]\eta=0. \tag{15}$$

Let us consider the function $\phi=X_iX^i$ on SM, we have:

$$g^{jk}D_kD_j\phi=g^{jk}\big[D_kX_iD_jX^i+X_iD_kD_jX^i+D_kX^iD_jX_i+X^iD_kD_jX_i\big]. \tag{16}$$

For a Landsberg manifold $D_kg_{ij}=0$, hence $X^iD_kD_jX_i=X_iD_kD_jX^i$, therefore (16) reduces to

$$g^{jk}D_kD_j\phi=2X_ig^{jk}D_kD_jX^i+2D^jX^iD_jX_i. \tag{17}$$

The left-hand side in (17) is divergence, hence by integration over SM and using divergence formula (7), we obtain:

$$\int_{SM}\big[X_ig^{jk}D_kD_jX^i+D^jX^kD_jX_k\big]\eta=0. \tag{18}$$

By subtracting (15) and (18) we get:

$$\int_{SM}\big[\big(X_ig^{jk}D_kD_jX^i-\big(R_{jk}+T^t_{rj}R^r_{0\ kt}\big)X^jX^k\big)+\big(D^jX^kD_jX_k-D_kX^jD_jX^k\big)+D_kX^kD_jX^j\big]\eta=0. \tag{19}$$

Note that $(R_{jk}+T^t_{rj}R^r_{0\ kt})X^jX^k=(R^i_k+T^{it}_rR^r_{0\ kt})X_iX^k$, hence relation (19) is equivalent to

$$\int_{SM} \left[X_i \left(g^{jk} D_k D_j X^i - \left(R^i_k + T^{it}_r R^r_{0\ kt} \right) X^k \right) + \frac{1}{2} \left(D^j X^k - D^k X^j \right) \left(D_j X_k - D_k X_j \right) + D_k X^k D_j X^j \right] \eta = 0. \tag{20}$$

By assumption, the first term in (20) is zero. The next two terms $\frac{1}{2}\|(D_j X_k - D_k X_j)\|^2$ and $(D_j X^j)^2$ are positive or zero, hence by (20) are zero. Therefore we obtain $D_j X_k - D_k X_j = 0$ and $D_j X^j = 0$, thus by definition X is a harmonic vector field on M. Conversely, let X be a harmonic vector field on a Landsberg manifold and $T^{it}_r R^r_{0\ kt} X^k = 0$. Replacing $D_i X_j = D_j X_i$ and $\frac{\partial X_i}{\partial y^j} = 0$ in (4) yields

$$D_i D_k X_j - D_k D_j X_i = -R^l_{j\ ik} X_l. \tag{21}$$

Multiplying (21) by g^{jk} on Landsberg manifold, we have:

$$D_i D^j X_j - g^{jk} D_k D_j X_i = -R^l_i X_l, \quad \text{or} \quad g^{jk} D_k D_j X^i = R^i_l X^l.$$

Hence proof is complete. □

Example 1. *Let (M, F) be a simply connected, compact Landsberg Manifold with positive constant flag curvature. It is well known that (M, F) reduces to a Riemannian manifold and $T^{it}_r = 0$, cf., [5] or [2] p. 129. Therefore a vector field X on M is harmonic if and only if $g^{jk} D_k D_j X^i = R^i_l X^l$ and $\frac{\partial X_i}{\partial y^j} = 0$.*

Corollary 3.2. *Let (M, F) be a compact Finsler manifold without boundary and with zero flag curvature. A necessary and sufficient condition for a vector field X on M to be harmonic is $g^{jk} D_k D_j X^i = 0$ and $\frac{\partial X_i}{\partial y^j} = 0$.*

Proof. It is well known that any compact Finsler manifold without boundary of zero flag curvature is a Landsberg manifold and *hh*-curvature of Berwald connection vanishes, cf. [2], p. 164 and [4], p. 328. Hence the proof is a direct conclusion of the above theorem. □

Theorem 3.3. *Let (M, F) be a compact Landsberg manifold without boundary and X a harmonic vector field on M. If $(R_{jk} + T^t_{rj} R^r_{0\ kt}) X^j X^k \geq 0$, then the covariant derivatives of X with respect to the Cartan and Berwald connections vanish. Moreover, there exists no non-zero harmonic vector field on (M, F) which satisfies the relation $(R_{jk} + T^t_{rj} R^r_{0\ kt}) X^j X^k > 0$.*

Proof. Let X be a harmonic vector field on M. By definition,

$$g^{ij} D_i X_j = 0, \qquad D_i X_j = D_j X_i, \qquad \frac{\partial X_i}{\partial y^j} = 0. \tag{22}$$

Therefore on a Landsberg manifold:

$$D_k X^k = g^{kj} D_k X_j = 0. \tag{23}$$

On the other hand by means of (22) we have:

$$D_k X^j D_j X^k = g^{jl} D_k X_l D_j X^k = g^{jl} D_l X_k D_j X^k = \|D_l X_k\|^2 \geq 0. \tag{24}$$

Substituting (23) and (24) in (15) we obtain:

$$\int_{SM} \left(\left(R_{jk} + T^t_{rj} R^r_{0\ kt} \right) X^j X^k + \|D_l X_k\|^2 \right) \eta = 0. \tag{25}$$

If the vector field X satisfies in $(R_{jk} + T^t_{rj} R^r_{0\ kt}) X^j X^k \geq 0$, then by (25) we find $\|D_l X_k\| = 0$ and hence $\nabla_l X_k = D_l X_k = 0$. Moreover, by (25) there exist no non-zero harmonic vector field on M which satisfies $(R_{jk} + T^t_{rj} R^r_{0\ kt}) X^j X^k > 0$. This completes the proof. □

Theorem 3.4. *Let (M, F) be an n-dimensional compact Landsberg manifold without boundary and X a harmonic vector field on M. If $(R_{jk} + T^t_{rj} R^r_{0\ kt}) X^j X^k \geq 0$, then the first de Rham cohomology group of SM satisfies $\dim H^1_{dR}(SM) \leq 2n - 1$. Moreover, if $(R_{jk} + T^t_{rj} R^r_{0\ kt}) X^j X^k > 0$, then $H^1_{dR}(SM) = 0$.*

Proof. Let X be a harmonic vector field on (M, F) endowed with a Berwald connection which satisfies $(R_{jk} + T^t_{rj} R^r_{0\ kt}) X^j X^k \geq 0$. By definition of harmonic vector fields and Theorem 3.3, we have $D_i X_j = D_{\dot{\partial}_i} X_j = 0$. Therefore the

1-form $X = X_i(z)\mathrm{d}x^i$ is parallel with respect to the Berwald connection on SM. It is well known that a parallel 1-form on a manifold is determined by its value at a point on the underlying manifold. Hence the dimension of the vector space of parallel 1-forms is at most equal to the dimension of the cotangent space $T_x^*(SM)$, that is $2n-1$. Therefore $\dim H^1_{\mathrm{d}R}(SM) \leq 2n-1$. Next, if $(R_{jk} + T^t_{rj} R^r_{0\ kt}) X^j X^k > 0$, then by means of Theorem 3.3 there is no non-zero harmonic vector field X which satisfies the above strict inequality. Hence SM has no nontrivial harmonic 1-form. Thus $H^1_{\mathrm{d}R}(SM) = 0$ and the proof is complete. □

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