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A generalization of Cramér large deviations for martingales

Une généralisation des grandes déviations de Cramér pour les martingales

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A B S T R A C T

In this note, we give a generalization of Cramér's large deviations for martingales, which can be regarded as a supplement of Fan et al. (2013) [3]. Our method is based on the change of probability measure developed by Grama and Haeusler (2000) [6].

R É S U M É

Dans cette note, nous donnons une généralisation des grandes déviations de Cramér pour les martingales, qui peut être considérée comme un supplément de Fan et al. (2013) [3]. Notre méthode est basée sur le changement de mesure de probabilité développé par Grama et Haeusler (2000) [6].

1. Introduction

Assume that $\eta_1, \ldots, \eta_n$ is a sequence of independent and identically distributed (i.i.d.) centered real valued random variables satisfying the following Cramér condition: $\mathbb{E}\exp\{c_0|\eta_1|\} < \infty$ for some $c_0 > 0$. Denote by $\sigma^2 = \mathbb{E}\eta_1^2$, $\xi_i = \eta_i/(\sqrt{n}\sigma)$ and $X_n = \sum_{i=1}^n \xi_i$. Cramér [1] has established the following asymptotic expansion of the tail probabilities of X_n, for all $0 \le x = o(n^{1/6})$ as $n \to \infty$,

$$\mathbb{P}(X_n > x) = \big(1 - \Phi(x)\big)\big(1 + o(1)\big), \tag{1}$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left\{-\frac{t^2}{2}\right\} dt$$

is the standard normal distribution function. More precise results can be found in Feller [4], Petrov [10,11], Saulis and Statulevičius [15], Sakhanenko [14] and [2], among others.

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Let $(\xi_i, \mathcal{F}_i)_{i=0,...,n}$ be a sequence of martingale differences defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\xi_0 = 0$ and $\{\emptyset, \Omega\} = \mathcal{F}_0 \subseteq ... \subseteq \mathcal{F}_n \subseteq \mathcal{F}$ are increasing σ-fields. Set

$$X_0 = 0, \qquad X_k = \sum_{i=1}^{k} \xi_i, \quad k = 1, ..., n. \tag{2}$$

Denote by $\langle X \rangle$ the quadratic characteristic of the martingale $X = (X_k, \mathcal{F}_k)_{k=0,...,n}$:

$$\langle X \rangle_0 = 0, \qquad \langle X \rangle_k = \sum_{i=1}^{k} \mathbb{E}\big(\xi_i^2 \mid \mathcal{F}_{i-1}\big), \quad k = 1, ..., n. \tag{3}$$

Consider the stationary case for simplicity. For the martingale differences having a $(2+p)$th moment, i.e. $\|\xi_i\|_{2+p} < \infty$ for some $p \in (0,1]$, expansions of the type (1) in the range $0 \le x = o(\sqrt{\log n})$ have been obtained by Haeusler and Joos [8], Grama [5], and Grama and Haeusler [7]. If the martingale differences are bounded $|\xi_i| \le C/\sqrt{n}$ and satisfy $\|\langle X \rangle_n - 1\|_\infty \le L^2/n$ a.s. for two positive constants C and L; expansion (1) has been firstly established by Račkauskas [12,13] in the range $0 \le x = o(n^{1/6})$, and then this range has been extended to a larger one $0 \le x = o(n^{1/4})$ by Grama and Haeusler [6] with a method based on a change of the probability measure. Recently, Fan et al. [3] have generalized the result of Grama and Haeusler [6] to a much larger range $0 \le x = o(n^{1/2})$ for ξ_i satisfying the following conditional Bernstein condition: for a positive constant C,

$$\big|\mathbb{E}\big(\xi_i^k \mid \mathcal{F}_{i-1}\big)\big| \le \frac{1}{2} k! \left(\frac{C}{\sqrt{n}}\right)^{k-2} \mathbb{E}\big(\xi_i^2 \mid \mathcal{F}_{i-1}\big) \quad \text{for all } k \ge 2 \text{ and all } 1 \le i \le n. \tag{4}$$

It is worth noting that the conditional Bernstein condition does not imply that ξ_i's are bounded.

The aim of this note is to extend the expansion of Fan et al. [3] to the case of martingale differences satisfying the following conditional Cramér condition considered in Liu and Watbled [9]:

$$\sup_i \mathbb{E}\big(\exp\big\{C_0 \sqrt{n} |\xi_i|\big\} \mid \mathcal{F}_{i-1}\big) \le C_1, \tag{5}$$

where C_0 and C_1 are two positive constants. It is worth noting that, in general, condition (5) does not imply the conditional Bernstein condition (4), unless $n\mathbb{E}(\xi_i^2|\mathcal{F}_{i-1})$ are all bounded from below by a positive constant. Thus our result is not a consequence of Fan et al. [3].

Throughout this paper, c and c_α, probably supplied with some indices, denote respectively a generic positive absolute constant and a generic positive constant depending only on α. Moreover, θ stands for any value satisfying $|\theta| \le 1$.

2. Main results

The following theorem is our main result, which can be regarded as a parallel result of Fan et al. [3] under the conditional Cramér condition:

(A1) $\sup_{1\le i\le n} \mathbb{E}(\exp\{c_0 n^{1/2} |\xi_i|\} | \mathcal{F}_{i-1}) \le c_1$;
(A2) $\|\langle X \rangle_n - 1\|_\infty \le \delta^2$ a.s., where δ is nonnegative and usually depends on n.

Theorem 2.1. *Assume conditions* (A1) *and* (A2). *Then there exists a positive absolute constant* α_0, *such that for all* $0 \le x \le \alpha_0 n^{1/2}$ *and* $\delta \le \alpha_0$, *the following equalities hold:*

$$\frac{\mathbb{P}(X_n > x)}{1 - \Phi(x)} = \exp\left\{\theta c_{\alpha_0}\left(\frac{x^3}{\sqrt{n}} + x^2\delta^2 + (1+x)\left(\frac{\log n}{\sqrt{n}} + \delta\right)\right)\right\} \tag{6}$$

and

$$\frac{\mathbb{P}(X_n < -x)}{\Phi(-x)} = \exp\left\{\theta c_{\alpha_0}\left(\frac{x^3}{\sqrt{n}} + x^2\delta^2 + (1+x)\left(\frac{\log n}{\sqrt{n}} + \delta\right)\right)\right\}, \tag{7}$$

where $|\theta| \le 1$. *In particular, for all* $0 \le x = o(\min\{n^{1/6}, \delta^{-1}\})$ *as* $\min\{n, \delta^{-1}\} \to \infty$,

$$\mathbb{P}(X_n \ge x) = \big(1 - \Phi(x)\big)\big(1 + o(1)\big). \tag{8}$$

From (6), we find that there is an absolute constant $\alpha_0 > 0$ such that, for all $0 \le x \le \alpha_0 n^{1/2}$ and $\delta \le \alpha_0$,

$$\left|\log \frac{\mathbb{P}(X_n > x)}{1 - \Phi(x)}\right| \le c_{\alpha_0}\left(\frac{x^3}{\sqrt{n}} + x^2\delta^2 + (1+x)\left(\frac{\log n}{\sqrt{n}} + \delta\right)\right). \tag{9}$$

Note that this result can be regarded as a refinement of the moderate deviation principle (MDP) under conditions (A1) and (A2). Let a_n be any sequence of real numbers satisfying $a_n \to \infty$ and $a_n n^{-1/2} \to 0$ as $n \to \infty$. If $\delta \to 0$ as $n \to \infty$, then inequality (9) implies the MDP for X_n with the speed a_n and good rate function $x^2/2$; for each Borel set B,

$$-\inf_{x\in B^o} \frac{x^2}{2} \le \liminf_{n\to\infty} \frac{1}{a_n^2} \log \mathbb{P}\left(\frac{1}{a_n} X_n \in B\right) \le \limsup_{n\to\infty} \frac{1}{a_n^2} \log \mathbb{P}\left(\frac{1}{a_n} X_n \in B\right) \le -\inf_{x\in \overline{B}} \frac{x^2}{2},$$

where B^o and $\overline{B}$ denote the interior and the closure of B, respectively (see Fan et al. [3] for details).

3. Sketch of the proof

Let $(\xi_i, \mathcal{F}_i)_{i=0,\dots,n}$ be a martingale differences satisfying condition (A1). For any real number λ with $|\lambda| \le c_0 n^{1/2}$, define

$$Z_k(\lambda) = \prod_{i=1}^{k} \frac{e^{\lambda\xi_i}}{\mathbb{E}(e^{\lambda\xi_i}|\mathcal{F}_{i-1})}, \quad k = 1, \dots, n, \ Z_0(\lambda) = 1.$$

Then $Z(\lambda) = (Z_k(\lambda), \mathcal{F}_k)_{k=0,\dots,n}$ is a positive martingale and for each real number λ with $|\lambda| \le c_0 n^{1/2}$ and each $k = 1, \dots, n$, the random variable $Z_k(\lambda)$ is a probability density on $(\Omega, \mathcal{F}, \mathbb{P})$. Thus we can define the *conjugate probability measure* $\mathbb{P}_\lambda$ on $(\Omega, \mathcal{F})$, where

$$\mathrm{d}\mathbb{P}_\lambda = Z_n(\lambda)\, \mathrm{d}\mathbb{P}. \tag{10}$$

Denote by $\mathbb{E}_\lambda$ the expectation with respect to $\mathbb{P}_\lambda$. Setting

$$b_i(\lambda) = \mathbb{E}_\lambda(\xi_i|\mathcal{F}_{i-1}) \quad \text{and} \quad \eta_i(\lambda) = \xi_i - b_i(\lambda) \quad \text{for } i = 1, \dots, n,$$

we obtain the decomposition of X_n similar to that of Grama and Haeusler [6]:

$$X_n = B_n(\lambda) + Y_n(\lambda), \tag{11}$$

where

$$B_n(\lambda) = \sum_{i=1}^{n} b_i(\lambda) \quad \text{and} \quad Y_n(\lambda) = \sum_{i=1}^{n} \eta_i(\lambda).$$

Note that $(Y_k(\lambda), \mathcal{F}_k)_{k=1,\dots,n}$ is also a sequence of martingale differences w.r.t. $\mathbb{P}_\lambda$.

In the sequel, we establish some auxiliary lemmas which will be used in the proof of Theorem 2.1. We first give upper bounds for the conditional moments.

Lemma 3.1. *Assume condition* (A1)*. Then*

$$\mathbb{E}\big(|\xi_i|^k \,\big|\, \mathcal{F}_{i-1}\big) \le k!\big(c_0 n^{1/2}\big)^{-k} c_1, \quad k \ge 3.$$

Proof. Applying the elementary inequality $x^k/k! \le e^x$ to $x = c_0|n^{1/2}\xi_i|$, we have, for $k \ge 3$,

$$|\xi_i|^k \le k!\big(c_0 n^{1/2}\big)^{-k} \exp\big\{c_0\big|n^{1/2}\xi_i\big|\big\}. \tag{12}$$

Taking conditional expectations on both sides of the last inequality, by condition (A1), we obtain the desired inequality. □

Remark 1. It is worth noting that both condition (A1) and the conditional Bernstein condition (4) imply the following hypothesis.

(A1′) There exists $\epsilon > 0$, usually depending on n, such that

$$\mathbb{E}\big(|\xi_i|^k \,\big|\, \mathcal{F}_{i-1}\big) \le c_1\, k!\epsilon^k \quad \text{for all } k \ge 2 \text{ and all } 1 \le i \le n.$$

When $\epsilon = c_2/\sqrt{n}$, condition (A1′), together with (A2), yields Theorem 2.1.

Using Lemma 3.1, we obtain the following two technical lemmas. Their proofs are similar to the arguments of Lemmas 4.2 and 4.3 of Fan et al. [3].

Lemma 3.2. *Assume conditions* (A1) *and* (A2)*. Then, for all* $0 \le \lambda \le \frac{1}{4}c_0 n^{1/2}$,

$$\big|B_n(\lambda) - \lambda\big| \le c\big(\lambda\delta^2 + \lambda^2 n^{-1/2}\big). \tag{13}$$

Lemma 3.3. *Assume conditions (A1) and (A2). Then, for all* $0 \le \lambda \le \frac{1}{4} c_0 n^{1/2}$,

$$\left|\Psi_n(\lambda) - \frac{\lambda^2}{2}\right| \le c\big(\lambda^2 \delta^2 + \lambda^3 n^{-1/2}\big),$$

where

$$\Psi_n(\lambda) = \sum_{i=1}^n \log \mathbb{E}\big(e^{\lambda \xi_i} \,\big|\, \mathcal{F}_{i-1}\big).$$

The following lemma gives the rate of convergence in the central limit theorem for the conjugate martingale $(Y_i(\lambda), \mathcal{F}_i)$ under the probability measure $\mathbb{P}_\lambda$. Its proof is similar to that of Lemma 3.1 of Fan et al. [3] by noting the fact that $\mathbb{E}(\xi_i^2 | \mathcal{F}_{i-1}) \le c/n$.

Lemma 3.4. *Assume conditions* (A1) *and* (A2)*. Then, for all* $0 \le \lambda \le \frac{1}{4} c_0 n^{1/2}$,

$$\sup_x \big|\mathbb{P}_\lambda\big(Y_n(\lambda) \le x\big) - \Phi(x)\big| \le c\left(\lambda \frac{1}{\sqrt{n}} + \frac{\log n}{\sqrt{n}} + \delta\right).$$

Proof of Theorem 2.1. The proof of Theorem 2.1 is similar to the arguments of Theorems 2.1 and 2.2 in Fan et al. [3] with $\epsilon = \frac{c_0}{4\sqrt{n}}$. However, instead of using Lemmas 4.2, 4.3 and 3.1 of [3], we shall make use of Lemmas 3.2, 3.3 and 3.4 respectively. □

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Appendix A

The proofs of Lemmas 3.2 and 3.3 are given below.

Proof of Lemma 3.2. Recall that $0 \le \lambda \le \frac{1}{4} c_0 n^{1/2}$. By the relation between $\mathbb{E}$ and $\mathbb{E}_\lambda$ on $\mathcal{F}_i$, we have

$$b_i(\lambda) = \frac{\mathbb{E}(\xi_i e^{\lambda \xi_i} | \mathcal{F}_{i-1})}{\mathbb{E}(e^{\lambda \xi_i} | \mathcal{F}_{i-1})}, \quad i = 1, \ldots, n.$$

Jensen's inequality and $\mathbb{E}(\xi_i | \mathcal{F}_{i-1}) = 0$ imply that $\mathbb{E}(e^{\lambda \xi_i} | \mathcal{F}_{i-1}) \ge 1$. Since

$$\mathbb{E}\big(\xi_i e^{\lambda \xi_i} \,\big|\, \mathcal{F}_{i-1}\big) = \mathbb{E}\big(\xi_i\big(e^{\lambda \xi_i} - 1\big) \,\big|\, \mathcal{F}_{i-1}\big) \ge 0,$$

by Taylor's expansion for e^x, we find that

$$B_n(\lambda) \le \sum_{i=1}^n \mathbb{E}\big(\xi_i e^{\lambda \xi_i} \,\big|\, \mathcal{F}_{i-1}\big) = \lambda \langle X \rangle_n + \sum_{i=1}^n \sum_{k=2}^{+\infty} \mathbb{E}\left(\frac{\xi_i (\lambda \xi_i)^k}{k!} \,\middle|\, \mathcal{F}_{i-1}\right). \tag{14}$$

Using Lemma 3.1, we obtain

$$\begin{aligned}
\sum_{i=1}^n \sum_{k=2}^{+\infty} \left|\mathbb{E}\left(\frac{\xi_i (\lambda \xi_i)^k}{k!} \,\middle|\, \mathcal{F}_{i-1}\right)\right| &\le \sum_{i=1}^n \sum_{k=2}^{+\infty} \big|\mathbb{E}\big(\xi_i^{k+1} \,\big|\, \mathcal{F}_{i-1}\big)\big| \frac{\lambda^k}{k!} \\
&\le \sum_{i=1}^n \sum_{k=2}^{+\infty} c_1\,(k+1)\lambda^k \big(c_0 n^{1/2}\big)^{-k-1} \\
&\le c_2\, \lambda^2 n^{-1/2}.
\end{aligned} \tag{15}$$

Condition (A2) together with (14) and (15) implies the upper bound of $B_n(\lambda)$:

$$B_n(\lambda) \le \lambda + \lambda \delta^2 + c_2\, \lambda^2 n^{-1/2}.$$

Using Lemma 3.1, we have

$$\begin{aligned}\mathbb{E}\big(e^{\lambda\xi_i}\mid\mathcal{F}_{i-1}\big) &\le 1+\sum_{k=2}^{+\infty}\left|\mathbb{E}\left(\frac{(\lambda\xi_i)^k}{k!}\,\middle|\,\mathcal{F}_{i-1}\right)\right| \\ &\le 1+\sum_{k=2}^{+\infty}c_1\,\lambda^k\big(c_0n^{1/2}\big)^{-k} \\ &\le 1+c_3\lambda^2n^{-1}. \end{aligned} \tag{16}$$

This inequality together with (15) and condition (A2) implies the lower bound of $B_n(\lambda)$:

$$\begin{aligned}B_n(\lambda) &\ge \left(\sum_{i=1}^n\mathbb{E}\big(\xi_ie^{\lambda\xi_i}\mid\mathcal{F}_{i-1}\big)\right)\big(1+c_3\,\lambda^2n^{-1}\big)^{-1} \\ &\ge \left(\lambda\langle X\rangle_n-\sum_{i=1}^n\sum_{k=2}^{+\infty}\left|\mathbb{E}\left(\frac{\xi_i(\lambda\xi_i)^k}{k!}\,\middle|\,\mathcal{F}_{i-1}\right)\right|\right)\big(1+c_3\,\lambda^2n^{-1}\big)^{-1} \\ &\ge \big(\lambda-\lambda\delta^2-c_2\,\lambda^2n^{-1/2}\big)\big(1+c_3\,\lambda^2n^{-1}\big)^{-1} \\ &\ge \lambda-\lambda\delta^2-c_4\,\lambda^2n^{-1/2}. \end{aligned}$$

The proof of Lemma 3.2 is finished. □

Proof of Lemma 3.3. Recall that $0\le\lambda\le\frac{1}{4}c_0n^{1/2}$. Since $\mathbb{E}(\xi_i|\mathcal{F}_{i-1})=0$, it is easy to see that

$$\Psi_n(\lambda)=\sum_{i=1}^n\left(\log\mathbb{E}\big(e^{\lambda\xi_i}\mid\mathcal{F}_{i-1}\big)-\lambda\mathbb{E}(\xi_i|\mathcal{F}_{i-1})-\frac{\lambda^2}{2}\mathbb{E}\big(\xi_i^2\mid\mathcal{F}_{i-1}\big)\right)+\frac{\lambda^2}{2}\langle X\rangle_n.$$

Using a two-term Taylor's expansion of $\log(1+x), x\ge0$, we obtain

$$\begin{aligned}\Psi_n(\lambda)-\frac{\lambda^2}{2}\langle X\rangle_n &= \sum_{i=1}^n\left(\mathbb{E}\big(e^{\lambda\xi_i}\mid\mathcal{F}_{i-1}\big)-1-\lambda\mathbb{E}(\xi_i|\mathcal{F}_{i-1})-\frac{\lambda^2}{2}\mathbb{E}\big(\xi_i^2\mid\mathcal{F}_{i-1}\big)\right) \\ &\quad -\frac{1}{2(1+|\theta|(\mathbb{E}(e^{\lambda\xi_i}|\mathcal{F}_{i-1})-1))^2}\sum_{i=1}^n\big(\mathbb{E}\big(e^{\lambda\xi_i}\mid\mathcal{F}_{i-1}\big)-1\big)^2. \end{aligned}$$

Since $\mathbb{E}(e^{\lambda\xi_i}|\mathcal{F}_{i-1})\ge1$, we find that

$$\begin{aligned}\left|\Psi_n(\lambda)-\frac{\lambda^2}{2}\langle X\rangle_n\right| &\le \sum_{i=1}^n\left|\mathbb{E}\big(e^{\lambda\xi_i}\mid\mathcal{F}_{i-1}\big)-1-\lambda\mathbb{E}(\xi_i|\mathcal{F}_{i-1})-\frac{\lambda^2}{2}\mathbb{E}\big(\xi_i^2\mid\mathcal{F}_{i-1}\big)\right| \\ &\quad +\frac{1}{2}\sum_{i=1}^n\big(\mathbb{E}\big(e^{\lambda\xi_i}\mid\mathcal{F}_{i-1}\big)-1\big)^2 \\ &\le \sum_{i=1}^n\sum_{k=3}^{+\infty}\frac{\lambda^k}{k!}\big|\mathbb{E}\big(\xi_i^k\mid\mathcal{F}_{i-1}\big)\big|+\frac{1}{2}\sum_{i=1}^n\left(\sum_{k=2}^{+\infty}\frac{\lambda^k}{k!}\big|\mathbb{E}\big(\xi_i^k\mid\mathcal{F}_{i-1}\big)\big|\right)^2. \end{aligned}$$

In the same way as in the proof of Lemma 3.2, by Lemma 3.1, we have

$$\left|\Psi_n(\lambda)-\frac{\lambda^2}{2}\langle X\rangle_n\right|\le c_3\lambda^3n^{-1/2}.$$

Combining this inequality with condition (A2), we obtain the desired inequality. □

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