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The first cohomology of separably rationally connected varieties

Le premier groupe de cohomologie des variétés séparablement, rationnellement connexes

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ABSTRACT

We present two proofs that for a smooth projective separably rationally connected variety over an algebraically closed field $H^1(X, \mathcal{O}_X) = 0$. The second, cohomological proof generalises to varieties admitting a free curve of higher genus.

RÉSUMÉ

Nous présentons deux démonstrations de la nullité de $H^1(X, \mathcal{O}_X)$ pour les variétés projectives, lisses, séparablement rationnellement connexes, sur un corps algébriquement clos. La seconde, cohomologique, se généralise aux variétés ayant une courbe libre de genre supérieur.

1. Introduction and statement

We call a variety rationally connected if there passes a rational curve through every two general points and separably rationally connected if there exists a morphism $f : \mathbb{P}^1 \to X$ such that f^*T_X is an ample vector bundle. In characteristic zero, these notions coincide, whereas they differ in characteristic p. Over an algebraically closed field of characteristic zero, a smooth projective separably rationally connected variety X has $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$ from Hodge theory (see [5, p. 249]). In a recent preprint, Biswas and dos Santos [1] prove a result that easily implies that in arbitrary characteristic, at least $H^1(X, \mathcal{O}_X) = 0$.

Theorem. *(See [1, Theorem 1.1].) Let X be a smooth projective separably rationally connected variety over k, an algebraically closed field. Let E be a vector bundle over X such that for each k-morphism $f : \mathbb{P}^1 \to X$, the pullback f^*E is trivial. Then E itself is trivial.*

The claim on vanishing of first cohomology can be seen as follows. Pick a class in $H^1(X, \mathcal{O}_X) = \mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_X)$ corresponding to a vector bundle E of rank two. After pulling back to any $f : \mathbb{P}^1 \to X$, we obtain:

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to f^*E \to \mathcal{O}_{\mathbb{P}^1} \to 0.$$

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It follows that f^*E is split since $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0$. Now from the main theorem in [1], E must itself be trivial. In positive characteristic we give another (cohomological) proof that at least $H^1(X, \mathcal{O}_X) = 0$, which is a special case of the following.

Theorem. *Let X be a smooth projective variety over an algebraically closed field k and $f : C \to X$ a morphism from a smooth projective curve such that f^*T_X is an ample bundle. Then $H^1(X, \mathcal{O}_X) = 0$.*

It should be noted that not much is known about the groups $H^i(X, \mathcal{O}_X)$ for smooth separably rationally connected varieties where $i > 1$ in positive characteristic. In the case of smooth Fano threefolds, Shepherd and Barron [12, Corollary 1.5] proved that $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$. It is also shown (ibid. Corollary 12.4) that at least in the case of Picard rank one, Fano threefolds are liftable to characteristic zero, so they are separably rationally connected (in general Fano varieties are only rationally chain connected) and hence satisfy the conditions of the theorem above. Smooth separably unirational (hence separably rationally connected) threefolds have been shown (see [11, Theorem 2.5]) to have $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, 2, 3$. In higher dimension, Kollár [8, Theorem 11] has shown that there exist singular Fano varieties in positive characteristic that are not even separably uniruled (see also [9, Chapter V]), yet although it is open whether smooth Fanos are all separably rationally connected, at least a general Fano hypersurface is so by [14, Theorem 1.4]. On the other hand, Fano varieties that are also liftable to $W_2(k)$ satisfy Kodaira vanishing by Deligne–Illusie and hence have $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$, but it is not known whether Fano varieties satisfy Kodaira vanishing (see [10, Remark 3.5]).

2. Proof of the theorem

For the proof of the main theorem, we proceed as follows. In a similar fashion to the case where $C = \mathbb{P}^1$, one proves $H^0(X, \Omega^m_X) = 0$ for $m > 0$ (see [6, Proposition 7.4]), essentially by noting that we can cover X by the images of embeddings (see [9, Theorem II.1.8]) from C where the restriction $T_X|_C$ is ample. Over $\mathbb{C}$, the theorem now follows as in the case of $\mathbb{P}^1$ from Hodge theory. Note that a theorem of Bogomolov and MacQuillan [2,7] in characteristic zero proves that the existence of a curve satisfying the conditions of the theorem implies the existence of a very free $f : \mathbb{P}^1 \to X$. In positive characteristic, however, this is not known, nor is it known that X is rationally connected (see [6] for a discussion in this direction). One can construct examples of $f : C \to X$ with f^*T_X ample by starting with a very free curve $\mathbb{P}^1 \to X$ and precomposing with a finite map $C \to \mathbb{P}^1$. In fact, in dimension three and above, a general deformation of such a morphism f will be an embedding (see [9, Theorem II.1.8]).

The main structure of our proof in positive characteristic follows mutatis mutandis from the proof of Theorem 2.1 in Nygaard's paper [11]. Consider the Artin–Schreier sequence of étale sheaves on X:

$$0 \to \mathbb{F}_p \to \mathbb{G}_a \xrightarrow{F-1} \mathbb{G}_a \to 0.$$

The cohomologies of $\mathbb{G}_a$ and $\mathcal{O}_X$ agree and since the latter is coherent, étale and Zariski cohomology agree, hence we may assume that all cohomology groups are taken in the étale site. We obtain an exact sequence

$$0 \to H^1(X, \mathbb{F}_p) \to H^1(X, \mathcal{O}_X) \xrightarrow{F-1} H^1(X, \mathcal{O}_X) \to 0$$

where the last map is surjective due to SGA7.XXII Proposition 1.2. Suppressing base points, we use a method of Suwa [13] to show that a p-group in the étale fundamental group $\pi_1(X)$ is trivial. In the case of $C = \mathbb{P}^1$, Kollár has proved that $\pi_1(X)$ is trivial using the de Jong–Starr Theorem, see [5, Corollaire 3.6] (also [1, Remark 2.5] for a correction), although in the case of higher genus C, the étale fundamental group could a priori be infinite (the author expects this is not the case however). Suwa, using a computation in crystalline cohomology, first proves that the vanishing of global differential forms implies that $h^i_p = \dim H^i(X, \mathbb{Q}_p) = 0$ for $i > 0$ from which $\chi_p(X) = \sum_i (-1)^i h^i_p = 1$. This result hence also holds in our setup. Note now that pulling back $f : C \to X$ under an étale cover $Y \to X$ gives a smooth projective curve (possibly of higher genus) $g : C' \to Y$ with g^*T_Y also ample. Now, let $\pi_1(X) \to G$ be any finite quotient, $Y \to X$ the finite étale cover corresponding to G and let $Y \to Z$ be the degree p^r subcover corresponding to a p-Sylow in G. From the discussion before, both Y and Z admit morphisms from curves whose pullback of the tangent bundle is ample and so have $\chi_p = 1$. By Crew's formula ([4], Corollary 1.7), $\chi_p(Y) = p^r \chi_p(Z)$ hence $p^r = \deg(Y/Z) = 1$. Hence we have obtained that there are no non-trivial étale covers of order dividing p (see [3] for a similar argument). Since maps $\pi_1(X) \to \mathbb{F}_p$ correspond to étale covers of order dividing p, we have $\mathrm{Hom}(\pi_1(X), \mathbb{F}_p) = 0$.

Now $H^1(X, \mathbb{F}_p) = \mathrm{Hom}(\pi_1(X), \mathbb{F}_p) = 0$ and by SGA7.XXII Corollaire 2.1, the semi-simple component of $H^1(X, \mathcal{O}_X)$ under the endomorphism induced by Frobenius F is isomorphic to $H^1(X, \mathbb{F}_p) \otimes k$, which is trivial. Hence F is nilpotent on $H^1(X, \mathcal{O}_X)$. The injectivity of the map of the corresponding sheaves induces $H^0(X, \mathcal{O}_X/F\mathcal{O}_X) \to H^0(X, \Omega^1_X) = 0$ and so from the cohomology of the short exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{F} \mathcal{O}_X \to \mathcal{O}_X/F\mathcal{O}_X \to 0$$

we obtain that $F : H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X)$ is injective. Since F is thus injective and nilpotent on first cohomology, the result follows.

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References

[1] I. Biswas, J.P.P. dos Santos, Triviality criteria for vector bundles over rationally connected varieties, Preprint, 2011.
[2] Feodor A. Bogomolov, Michael L. MacQuillan, Rational curves on foliated varieties, IHES, Preprint, 2001.
[3] A. Chambert-Loir, Points rationnels et groupes fondamentaux: applications de la cohomologie p-adique (d'après P. Berthelot, T. Ekedahl, H. Esnault, etc.), Astérisque 294 (2004) 125–146.
[4] R.M. Crew, Étale p-covers in characteristic p, Compos. Math. 52 (1) (1984) 31–45.
[5] O. Debarre, Variétés rationnellement connexes (d'après T. Graber, J. Harris, J. Starr et A.J. de Jong), in: Séminaire Bourbaki, vols. 2001/2002, Astérisque 290 (2003) (2003) 243–266.
[6] F. Gounelas, Free curves on varieties, Preprint, arXiv:1208.4055, 2012.
[7] S. Kebekus, L. Solá Conde, M. Toma, Rationally connected foliations after Bogomolov and McQuillan, J. Algebr. Geom. 16 (1) (2007) 65–81.
[8] J. Kollár, Nonrational hypersurfaces, J. Amer. Math. Soc. 8 (1) (1995) 241–249.
[9] J. Kollár, Rational Curves on Algebraic Varieties, Ergeb. Math. Ihrer Grenzgeb. 3. Folge (A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]), vol. 32, Springer-Verlag, Berlin, 1996.
[10] J. Kollár, Singularities of the Minimal Model Program, Cambridge Tracts in Mathematics, vol. 200, Cambridge University Press, Cambridge, UK, 2013, with a collaboration of Sándor Kovács.
[11] N. Nygaard, On the fundamental group of a unirational 3-fold, Invent. Math. 44 (1) (1978) 75–86.
[12] N.I. Shepherd-Barron, Fano threefolds in positive characteristic, Compos. Math. 105 (3) (1997) 237–265.
[13] Noriyuki Suwa, A note on the fundamental group of a unirational variety, Proc. Jpn. Acad., Ser. A, Math. Sci. 59 (3) (1983) 98–99.
[14] Yi Zhu, Fano hypersurfaces in positive characteristic, Preprint, 2011.