C. R. Acad. Sci. Paris, Ser. I 352 (2014) 889–893

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Complex analysis/Partial differential equations

On the higher dimensional harmonic analog of the Levinson log log theorem

Sur l'analogue harmonique du théorème log log *de Levinson pour plusieurs dimensions*

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A R T I C L E I N F O

Article history:
Received 6 August 2014
Accepted after revision 23 September 2014
Available online 1 October 2014

Presented by Jean-Pierre Kahane

A B S T R A C T

Let M: $(0,1) \to [e,+\infty)$ be a decreasing function such that $\int_0^1 \log\log M(y)\,dy < +\infty$. Consider the set $\mathcal{H}_M$ of all functions u harmonic in $P := \{(x,y): x \in \mathbb{R}^{n-1}, y \in \mathbb{R}, |x| < 1, |y| < 1\}$ and satisfying $|u(x,y)| \le M(|y|)$. We prove that $\mathcal{H}_M$ is a normal family in P.

R É S U M É

Soit M: $(0,1) \to [e,+\infty)$ une fonction décroissante telle que $\int_0^1 \log\log M(y)\,dy < +\infty$. Considérons l'ensemble H_M de toutes les fonctions u qui sont harmoniques dans $P := \{(x,y) \in \mathbb{R}^n : x \in \mathbb{R}^{n-1}, y \in \mathbb{R}, |x| < 1, |y| < 1\}$ et satisfont $|u(x,y)| \le M(|y|)$. On montre que H_M est une famille normale dans P.

Let P be a rectangle $(-a,a) \times (-b,b)$ in $\mathbb{R}^2$ and let $M : (0,b) \to [e,+\infty)$ be a decreasing function. Consider the set $\mathcal{F}_M$ of all functions f holomorphic in P such that $|f(x,y)| \le M(|y|)$, $(x,y) \in P$. The classical Levinson theorem asserts that $\mathcal{F}_M$ is a normal family in P if $\int_0^b \log\log M(y)\,dy < +\infty$. We refer the reader to [4–8,13,14,16,17,19,20,22–24] for various proofs, history of the question and related topics. This statement is sharp, i.e. for regular (continuous and decreasing) majorants M, the family $\mathcal{F}_M$ is normal if and only if $\int_0^b \log\log M(y)\,dy < +\infty$ (see [16], pp. 379–383 and [4]).

The function $\log^+ x$ is defined by $\log^+ x = \begin{cases} \log x, & x \ge 1 \\ 0, & x \le 1 \end{cases}$. Our result is the following theorem, which extends the Levinson log log theorem for holomorphic functions to harmonic functions in $\mathbb{R}^n$, $n \ge 2$.

Theorem 0.1. *Let Ω denote the set $\{(x,y) : x \in \mathbb{R}^{n-1}, y \in \mathbb{R}, |x| < R, |y| < H\}$, where R and H are some positive numbers. Suppose a function $M : (0,H) \to \mathbb{R}_+$ is decreasing and*

$$\int_0^H \log^+ \log^+ M(y)\,dy < +\infty. \tag{1}$$

Then the set $\mathcal{H}_M$ of all functions u harmonic in Ω and satisfying $|u(x,y)| \le M(|y|)$, $(x,y) \in \Omega$, is uniformly bounded on any compact subset of Ω.

http://dx.doi.org/10.1016/j.crma.2014.09.019
1631-073X/

This result has been proved by Dyn'kin in [8] by a different method under some stronger regularity conditions imposed on M. For any compact set $K \subset \Omega$, our approach provides an explicit estimate for $\sup_{u \in \mathcal{H}_M} \sup_K |u|$ in terms of M, K and Ω. We obtain Theorem 0.1 as a corollary of the "holomorphic" Levinson theorem by a reduction to axially symmetric functions u. First, we prove Theorem 0.1 in dimension 4, which implies the 3-dimensional case. Then we reduce the case of odd n to the case $n = 3$. The case of even n follows by adding a dummy variable. The main obstacle, which appears in the higher-dimensional harmonic analog of the Levinson log log theorem, is the fact that $\log |\nabla u|$ is not necessarily subharmonic for a general harmonic function u in $\mathbb{R}^n$ if $n \geq 3$.

Some of the proofs of the "holomorphic" Levinson log log theorem are of a complex nature, some use implicitly or explicitly harmonic measure estimates in cusp-like domains, but most of the proofs require the monotonicity condition on M, except for the brilliant idea due to Domar (see [6,7,16]), which avoids any regularity assumptions on M, even the monotonicity. We will sketch Domar's proof in Section 1, and use it to obtain explicit uniform estimates for $\mathcal{H}_M$ in higher dimensions.

Let $d(x, y)$ denote the Euclidean distance between x and y in $\mathbb{R}^n$. For any $X, Y \subset \mathbb{R}^n$ put $d(X, Y) := \inf\{d(x, y) : x \in X, y \in Y\}$. The symbol λ_n will denote the n-dimensional Lebesgue measure.

1. Domar's argument

Theorem 1.1. *Let f be a holomorphic function in a rectangle $P := (-a, a) \times (-b, b)$. Suppose that a function $M(y)$ satisfies $\int_{-b}^{b} \log^+ \log^+ M(y)\,dy < +\infty$ and $|f(x + iy)| \leq M(y)$ for all $(x, y) \in P$. Then for any compact set $K \subset P$, there exists a constant $C = C(M, d(K, \partial P))$ such that $\sup_K |f| < C$.*

Theorem 1.1 immediately follows from the next lemma on subharmonic functions, since $\log |f|$ is subharmonic.

Lemma 1.2. *Let v be a subharmonic function in a rectangle $P := (-a, a) \times (-b, b)$. Suppose that a function $\tilde{M}$ satisfies $\int_{-b}^{b} \log^+ \tilde{M}(y)\,dy < +\infty$ and $v(x + iy) \leq \tilde{M}(y)$ for all $(x, y) \in P$. Then for any compact set $K \subset P$, there exists a constant $C = C(\tilde{M}, d(K, \partial\Omega))$ such that $\sup_K v \leq C$.*

Sketch of the proof. Let $F(t) := \lambda_1(\{y \in (-b, b) \colon \tilde{M}(y) \geq t\})$ denote the complementary cumulative distribution function of $\tilde{M}(y)$. The logarithmic integral condition $\int_{-b}^{b} \log^+ \tilde{M}(y)\,dy < +\infty$ can be reformulated in terms of F, namely $\sum_{i=0}^{+\infty} F(2^i) < +\infty$ if $\int_{-b}^{b} \log^+ \tilde{M}(y)\,dy < +\infty$ (see [16], pp. 378–379). Then there exists a positive number C such that

$$\sum_{i=-1}^{+\infty} F\left(2^i C\right) < \frac{\pi}{8} d(K, \partial P). \tag{2}$$

Our aim is to show that $\sup_K v \leq C$. Assume the contrary. Suppose there is $z_0 \in K$ with $v(z_0) > C$. Let A_t denote the set $\{z \in P : v(z) \geq t\}$.

Proposition 1.1. *If a point $z \in P$ satisfies $v(z) \geq \mathcal{C}$ with $\mathcal{C} > 0$, and $d(z, \partial P) > \frac{8}{\pi} F(\mathcal{C}/2)$, then there is a $\zeta \in P$ such that $d(z, \zeta) \leq \frac{8}{\pi} F(\mathcal{C}/2)$ and $v(\zeta) \geq 2\mathcal{C}$.*

Consider the ball B centered at z with radius $r = \frac{8}{\pi} F(\mathcal{C}/2)$. Note that $B \subset P$, since $d(z, \partial P) > \frac{8}{\pi} F(\mathcal{C}/2)$. Now, the subharmonicity of v will be exploited:

$$\mathcal{C} \leq v(z) \leq \frac{1}{\lambda_2(B)} \int_B v = \frac{1}{\lambda_2(B)} \left(\int_{B \setminus A_{\mathcal{C}/2}} v + \int_{B \cap A_{\mathcal{C}/2}} v \right) \leq \mathcal{C}/2 + \frac{1}{\lambda_2(B)} \int_{B \cap A_{\mathcal{C}/2}} v.$$

Hence

$$\begin{aligned} \mathcal{C}/2 &\leq \frac{1}{\lambda_2(B)} \int_{B \cap A_{\mathcal{C}/2}} v \leq \frac{1}{\pi r^2} \sup_B v \cdot \lambda_2(B \cap A_{\mathcal{C}/2}) \\ &\leq \frac{1}{\pi r^2} \sup_B v \cdot \lambda_1\left(\left\{x \mid \exists y : (x, y) \in B \cap A_{\mathcal{C}/2}\right\}\right) \cdot \lambda_1\left(\left\{y : \exists x : (x, y) \in B \cap A_{\mathcal{C}/2}\right\}\right) \\ &\leq \frac{1}{\pi r^2} \sup_B v \cdot 2r F(\mathcal{C}/2) = \frac{1}{4} \sup_B v. \end{aligned}$$

Thus $2C \leq \sup_B v$ and the proposition is proved.

Using the proposition and taking z_0 in place of z and C in place of $\mathcal{C}$, we obtain a point z_1 such that $v(z_1) \geq 2C$ and $d(z_1, z_0) \leq \frac{8}{\pi} F(C/2)$. Recall that $d(z_0, \partial P) > \frac{8}{\pi} \sum_{i=-1}^{+\infty} F(2^i C)$, hence $d(z_1, \partial P) > \frac{8}{\pi} \sum_{i=0}^{+\infty} F(2^i C)$. Exploiting the proposition infinitely many times, we obtain a sequence $\{z_i\}_{i=0}^{+\infty}$ such that $v(z_i) \ge 2^i C$ and $d(z_i, z_{i+1}) \le \frac{8}{\pi} F(2^{i-1}C)$. By (2) $\{z_i\}$ has a limit point $z \in P$, hence $v(z) \ge \lim_{i\to\infty} v(z_i) = +\infty$, and a contradiction has been obtained. □

Remark 1. Domar's argument also provides explicit estimates in Theorem 1.1 of $C(M, d(K, \partial P))$. Put $F(t) := \lambda_1(\{y : \log^+ M(y) \ge t\})$. If $C > 0$ and $d(K, \partial P) > \frac{8}{\pi} \sum_{i=-1}^{+\infty} F(2^i C)$, then $|f| \le \exp(C)$ on K.

2. Axially symmetric harmonic functions

Consider $\mathbb{R}^n = \{x = (x_1, \dots, x_n) : x_i \in \mathbb{R}\}$. By ρ we denote $\sqrt{\sum_{i=1}^{n-1} x_i^2}$ and $h := x_n$. A function u defined in $\mathbb{R}^n$ is called axially symmetric if $u = u(\rho, h)$, i.e. u is invariant under orthogonal transformations of the first $(n-1)$ coordinates. An axially symmetric harmonic function u satisfies the following equation (equation for the axially symmetric potentials):

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{\partial^2 u}{\partial h^2} + \frac{n-2}{\rho}\frac{\partial u}{\partial \rho} = 0. \tag{3}$$

We are going to use two ideas. The first one reduces axially symmetric harmonic functions in $\mathbb{R}^4$ to ordinary harmonic functions in $\mathbb{R}^2$. The second trick reduces axially symmetric harmonic functions in $\mathbb{R}^{2k+3}$ to harmonic functions in $\mathbb{R}^3$. It will help in dimension $n \ge 5$. We refer the reader to [1,9,10,15,21,25] and references therein, where these and related ideas appear in a different context. However, we are not able to locate their origin.

2.1. From $\mathbb{R}^4$ to $\mathbb{R}^2$

Suppose u is an axially symmetric harmonic function in an axially symmetric domain $\Omega \subset \mathbb{R}^4$. Consider the set $\tilde{\Omega}_+ \subset \mathbb{R}^2$ defined by $x \in \Omega \iff (\rho(x), h(x)) \in \tilde{\Omega}_+$. It is easy to see from (3) that the function

$$\tilde{u}(\rho, h) = \rho u\big(|\rho|, h\big) \tag{4}$$

is harmonic in Int $\tilde{\Omega}_+$. Define $\tilde{\Omega}_-$ by $x \in \Omega \iff (-\rho(x), h(x)) \in \tilde{\Omega}_-$. Let $\tilde{\Omega}$ be the union of $\tilde{\Omega}_+$ and $\tilde{\Omega}_-$. Then $\tilde{\Omega}$ is a domain in $\mathbb{R}^2$, symmetric with respect to the line $\rho = 0$. By the Schwarz reflection principle, we see that (4) defines an odd (with respect to ρ) harmonic function in $\tilde{\Omega}$.

2.2. From $\mathbb{R}^{2k+3}$ to $\mathbb{R}^3$

Let $u = u(\rho, h)$ be an axially symmetric harmonic function in $\mathbb{R}^{2k+3}$. Put

$$v(\varphi, \rho, h) = \rho^k e^{ik\varphi} u(\rho, h), \tag{5}$$

where (φ, ρ, h) are cylindrical coordinates in $\mathbb{R}^3$. Then v is a harmonic (complex-valued) function in $\mathbb{R}^3$. Indeed, $\Delta v = \frac{\partial^2 v}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial v}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 v}{\partial \varphi^2} + \frac{\partial^2 v}{\partial h^2} = 0 + \rho^k e^{ik\varphi}(\frac{\partial^2 u}{\partial \rho^2} + \frac{\partial^2 u}{\partial h^2} + \frac{2k+1}{\rho}\frac{\partial u}{\partial \rho}) = 0$. The last argument shows that v is harmonic in $\mathbb{R}^3 \setminus \{\rho = 0\}$. Note that v is continuous up to the line $\{\rho = 0\}$, which is a removable singularity for bounded harmonic functions (see [2], p. 200). Thus v is harmonic in $\mathbb{R}^3$.

3. Proof of Theorem 0.1

Proof of the case $n = 4$. Fix $\varepsilon > 0$: $R, H > \varepsilon$. Take any $x_0 \in \mathbb{R}^{n-1}$ with $|x_0| < R - \varepsilon$. Consider any function u from $\mathcal{H}_M$. It is sufficient to show that there is $C = C(M, H, \varepsilon)$ such that $|u(x_0, h)| \le C$ for any h: $|h| < H - \varepsilon$. Denote the set $\{(x, y) : x \in \mathbb{R}^{n-1}, y \in \mathbb{R}, |x| < \varepsilon, |y| < H\}$ by P_ε and consider the function $\tilde{u} : P_\varepsilon \to \mathbb{R}$ defined by $\tilde{u}(x, y) = u(x - x_0, y)$. Note that $|\tilde{u}(x, y)| \le M(|y|)$ on P_ε.

Let us make an axial symmetrization step. Denote by $O(3)$ the group of orthogonal transformations in $\mathbb{R}^3$, let dS be the Haar measure on $O(3)$. For any $g \in O(3)$ we use the notation $\tilde{u}_g$ for the function $\tilde{u}(gx, y)$. It is clear that $\tilde{u}_g$ is harmonic in P_ε, $\tilde{u}_g(0, y) = \tilde{u}(0, y) = u(x_0, y)$ and $|\tilde{u}_g(x, y)| \le M(|y|)$ on P_ε. Put $w(x, y) := \int_{O(3)} \tilde{u}_g(x, y)\, dS(g)$, $(x, y) \in P_\varepsilon$, it is evident that w also enjoys the properties from the preceding sentence and $w = w(\rho, h)$ is axially symmetric. We have reduced the 4-dimensional case to the following lemma.

Lemma 3.1. *Suppose $w = w(\rho, h)$ is an axially symmetric harmonic function in the truncated cylinder P_ε and $|w(x, y)| \le M(|y|)$. Then there is a constant $C = C(M, H, \varepsilon)$ such that $|w(0, y)| < C$ for any $y \in (-H + \varepsilon, H - \varepsilon)$.*

Proof. Put $v(\rho, h) := \rho w(|\rho|, h)$. By Section 2.1, v is harmonic in $(-\varepsilon, \varepsilon) \times (-H, H)$. Denote $\rho + ih$ by ζ and $\frac{\partial v}{\partial \rho} - i\frac{\partial v}{\partial h}$ by f. Then f is a holomorphic function in $(-\varepsilon, \varepsilon) \times (-H, H)$. Denote the set $(-\varepsilon/2, \varepsilon/2) \times (-H + \varepsilon/2, H - \varepsilon/2)$ by $\tilde{P}_{\varepsilon/2}$.

Take any $\zeta = (\rho, h) \in \tilde{P}_{\varepsilon/2}$ with $h \leq \varepsilon$ and consider a disk $B_{h/2}(\zeta) := \{z : |z - \zeta| < h/2\}$. Since $|v(\rho, h)| \leq M(|h|)$ and M is decreasing $\sup\{|v|(x): x \in B_{h/2}(\zeta)\} \leq M(h/2)$. Applying standard Cauchy's estimates of derivatives of harmonic functions, we obtain $|\nabla v|(\zeta) \leq C_1 \frac{\sup\{|v|(x): x \in B_{h/2}(\zeta)\}}{h/2} \leq C_2 \frac{M(h/2)}{h}$. By C_1, C_2, C_3 we will denote absolute constants, whose value is less than 100. We note that $|f| = |\nabla v|$. Hence $|f|(\zeta) \leq C_2 \frac{M(h/2)}{h}$.

If $\zeta \in \tilde{P}_{\varepsilon/2}$ with $h \geq \varepsilon$, then $B_{\varepsilon/4}(\zeta) \subset (-\varepsilon, \varepsilon) \times (-H, H)$. Using in a similar way Cauchy's estimates, we obtain $|f(\zeta)| \leq C_3 \frac{M(h/2)}{\varepsilon}$. We therefore have $|f(\zeta)| \leq \max(\frac{100}{\varepsilon}, \frac{100}{h})M(h/2)$ for any $\zeta \in \tilde{P}_{\varepsilon/2}$. Denote $\max(\frac{100}{\varepsilon}, \frac{100}{h})M(h/2)$ by $\tilde{M}(h)$. It follows from the inequality $\log^+ a + \log^+ b + \log 2 \geq \log^+(a+b)$ that $\int_{-H}^{H} \log^+ \log^+ M(y)\,dy < +\infty$ implies $\int_{-H+\varepsilon/2}^{H-\varepsilon/2} \log^+ \log^+ \tilde{M}(y)\,dy < +\infty$.

Now, we are in a position to apply Theorem 1.1 to the function f holomorphic in $\tilde{P}_{\varepsilon/2}$ with the majorant $\tilde{M}$, that gives us a positive constant $C = C(M, H, \varepsilon)$: $|f(0, h)| < C$ for $h \in (-H + \varepsilon, H - \varepsilon)$. Recalling that $v(\rho, h) = \rho w(\rho, h)$, this yields $|w(0, h)| = |v_\rho(0, h)| \leq |f(0, h)| \leq C(M, H, \varepsilon)$.

Remark 2. Let $\tilde{F}(t)$ denote $\lambda_1(\{h \in (-H + \varepsilon/2, H - \varepsilon/2) : \max(\frac{100}{\varepsilon}, \frac{100}{h})M(h/2) \geq \exp(t)\})$. Then $C(M, H, \varepsilon)$ can be given explicitly in terms of $\tilde{F}$ in view of Remark 1. Namely, if $\varepsilon/2 > \frac{8}{\pi} \sum_{i=-1}^{+\infty} \tilde{F}(2^i C)$ for a positive constant C, then $u(x, y) \leq \exp(C)$ for all (x, y) with $|x| \leq R - \varepsilon$, $|h| \leq H - \varepsilon$.

Remark 3. The 4-dimensional case of Theorem 0.1 implies the 3-dimensional one (as well as the 2-dimensional), because we can always add a dummy coordinate to $\mathbb{R}^3$.

Proof of the case $n \geq 5$. We will consider only the case of odd $n = 2k + 3$. Now, we know that Theorem 0.1 holds for $n = 2, 3, 4$. We will prove the case of odd $n = 2k + 3$ reducing it to the case $n = 3$ with the help of the idea discussed in Section 2.2. The case of even n follows immediately. As in the proof of 4-dimensional case we can perform the axial-symmetrization step and Theorem 0.1 is reduced to the following lemma.

Lemma 3.2. *Suppose $u = u(\rho, h)$ is an axially symmetric harmonic function in a truncated cylinder $P_\varepsilon = \{(x \in \mathbb{R}^{n-1}, y \in \mathbb{R}, |x| < \varepsilon, |y| < H)\}$ such that $|u(x, y)| \leq M(|y|)$. Then there is a constant $\mathcal{C} = \mathcal{C}(n, M, H, \varepsilon)$ such that $|u(0, y)| < \mathcal{C}$ for $y \in (-H + \varepsilon, H - \varepsilon)$.*

Following Section 2.2 we consider a function v defined by $v(\varphi, \rho, h) = \mathrm{Re}(\rho^k e^{ik\varphi} u(\rho, h))$ on the set $\{\varphi \in [0, 2\pi), \rho \in [0, \varepsilon), h \in (-H + \varepsilon, H + \varepsilon)\}$, where v is harmonic. With the help of the 3-dimensional case of Theorem 0.1, we can obtain $|v(\varphi, \rho, h)| < C(M, H, \varepsilon/2)$ for $\varphi \in [0, 2\pi), \rho \in [0, \varepsilon/2), h \in (-H + \varepsilon/2, H - \varepsilon/2)$. Then for any $h \in (-H + \varepsilon, H - \varepsilon)$ and the ball B centered at the point $(0, 0, h)$ with radius $\varepsilon/2$ we have $\sup_B |v| \leq C(M, H, \varepsilon/2)$. Applying standard estimates of the higher derivatives of harmonic functions we obtain $\frac{\partial^k}{\partial \rho^k} v \leq C(k) \frac{C(M, H, \varepsilon/2)}{(\varepsilon/2)^k}$ on the set $\{\varphi \in [0, 2\pi), \rho = 0, h \in (-H + \varepsilon/2, H - \varepsilon/2)\}$, where $C(k)$ is a constant depending only on dimension $(n = 2k + 3)$. Take $\varphi = \rho = 0$ and see that $\frac{\partial^k v}{\partial \rho^k}(0, 0, h) = k! u(0, h)$. Thus $|u(0, h)| \leq C(k) \frac{C(M, H, \varepsilon/2)}{(\varepsilon/2)^k}$ for $h \in (-H + \varepsilon, H + \varepsilon)$. □

Question on one-sided estimates. Suppose that z_0 is a point in a square $Q = (-1, 1) \times (-1, 1)$ and M is a positive (decreasing and regular) function on $(0, 1)$. Under what assumptions on M is the family F_M^+ of all functions f holomorphic in Q and satisfying $\mathrm{Im}(f(z)) \leq M(|\mathrm{Im}(z)|)$, $f(z_0) = 0$ normal in Q?

4. Application to the universal polynomial expansions of harmonic functions

Consider the unit ball $\mathbb{B} := B_1(0)$ in $\mathbb{R}^n$. Any function h harmonic in $\mathbb{B}$ admits a power series expansion $h = \sum_{n=0}^{+\infty} h_n$, where h_n is a homogeneous harmonic polynomial of degree n. It is said that h belongs to the collection U_H, of harmonic functions in B with universal homogeneous polynomial expansions, if for any compact set $K \subset \mathbb{R}^n \setminus \mathbb{B}$ with connected complement and any harmonic function u in a neighborhood of K, there is a subsequence $\{N_k\}$ of $\mathbb{N}$ such that $\sum_0^{N_k} h_n \to u$ uniformly on K. This class of universal functions has been studied in [3,11,12,18]. The following statement improves Theorem 7 from [11] on the boundary behavior of functions from U_H.

Theorem 4.1. *Let $\psi : [0, 1) \to \mathbb{R}^+$ be an increasing function such that $\int_0^1 \log^+ \log^+ \psi(t)\,dt < +\infty$. If $h = \sum_{n=0}^{+\infty} h_n$ enjoys $|h(x)| \leq \psi(|x|)$ on $B_r(\omega) \cap \mathbb{B}$ for some $\omega \in \partial\mathbb{B}$ and $r > 0$, then $f \notin U_H$.*

We won't prove Theorem 4.1 here, because all necessary ingredients of the proof with one exception are given in [11], where Theorem 4.1 is proved under the stronger assumption $\int_0^1 \log^+ \psi(t)\,dt < +\infty$ in place of $\int_0^1 \log^+ \log^+ \psi(t)\,dt < +\infty$. The only missing ingredient in [11], which allows us to replace one log by log log, is the "harmonic" analog of the Levinson log log theorem in higher dimensions (its version in a ball, which follows from Theorem 0.1 with the help of the Kelvin transform).

Acknowledgements

We are grateful to D. Khavinson for explaining the application of the "harmonic" analog of the Levinson log log theorem to the boundary behavior of the universal power expansions of harmonic functions.

This research was sponsored by the grant of the Russian Science Foundation (project No. 14-21-00035).

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