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Algebra/Homological algebra

Some properties of the extremal algebras

Quelques propriétés des algèbres extrémales

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ABSTRACT

The (generalized) extremal algebra [4] is Noetherian, Auslander regular and Cohen–Macaulay. A necessary and sufficient condition is given for the generalized extremal algebras being Calabi–Yau. The point modules over these algebras are described explicitly.

RÉSUMÉ

En nous appuyant sur les travaux de Fløystad et Vatne, nous décrivons quelques propriétés homologiques des algèbres extrémales. Plus précisément, nous montrons que les algèbres extrémales sont intègres, nœthériennes, régulières au sens d'Auslander, de Cohen–Macaulay et de Calabi–Yau. Nous calculons également les modules cycliques de la série de Hilbert $(1-t)^{-1}$ sur ces algèbres extrémales.

0. Introduction

Enveloping algebras of graded Lie algebras form a basic class of Artin–Schelter regular algebras. A complete proof of this fact is given in [4, Theorem 2.1]. Floystad and Vatne prove that the enveloping algebra of any positive graded Lie algebra is Artin–Schelter regular with the global dimension and Gelfand–Kirillov dimension equal to the dimension of the Lie algebra as a vector space. They also construct a 5-dimensional Artin–Schelter algebra, called the extremal algebra. They show that the Hilbert series of the extremal algebra cannot be realized as the one of the enveloping algebra of any graded Lie algebra generated in degree one.

After recalling the definition given by Floystad and Vatne in [4], it is proved in Section 1 that the (generalized) extremal algebra is Noetherian, Auslander regular and Cohen–Macaulay. A necessary and sufficient condition is also given there for the generalized extremal algebras being Calabi–Yau. The point modules over the generalized extremal algebras are described in Section 2.

The base field k is assumed to be algebraically closed of characteristic zero. All vector spaces and algebras are over k. All the algebras are graded algebras generated in degree one.

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1. Extremal algebras

Let $\mathcal{A}$ be the extremal algebra given in [4]. Namely, $\mathcal{A}$ is the quotient algebra of the free algebra $k\langle x, y\rangle$ modulo the following three commutator relations:

$$\left[x^2, y\right], \qquad \left[x, y^3\right], \qquad [x, yRy], \quad \text{where } R \triangleq yxyx + xy^2x + xyxy.$$

Floystad and Vatne construct the minimal projective resolution of the trivial module ${}_{\mathcal{A}}k$ and prove that $\mathcal{A}$ is a 5-dimensional Artin–Schelter regular algebra [4, Theorem 4.2]. Moreover,

$$\left\{y^{n_y} B^{n_B} C^{n_C} A^{n_A} x^{n_x} \,\middle|\, n_y, n_B, n_C, n_A, n_x \in \mathbb{N}\right\}$$

is a k-linear basis for $\mathcal{A}$ where $A \triangleq xy$, $B \triangleq xy^2$, $C \triangleq AB = xyxy^2$.

Proposition 1.1. *The algebra $\mathcal{A}$ is strongly Noetherian, Auslander regular and Cohen–Macaulay.*

Proof. Let $z_1 = x^2$ and $z_2 = y^3$. Then z_1 and z_2 are normal (in fact, central) elements of $\mathcal{A}$. So, the image of z_2 is central in $\mathcal{A}_1 = \mathcal{A}/(z_1)$. Let $\mathcal{A}_2 = \mathcal{A}_1/(z_2) \cong k\langle x, y\rangle/(x^2, y^3, [x, yRy])$. Since $x(xy + yx) \equiv (xy + yx)x \bmod (x^2)$ and $R \equiv (xy + yx)^2 \bmod (x^2)$, then $x(xy + yx)^3 \equiv (xy + yx)^3 x \bmod (x^2)$ and

$$y(xy + yx)^3 - (xy + yx)^3 y \equiv yR(xy + yx) - (xy + yx)Ry \equiv -[x, yRy] \bmod \left(x^2\right).$$

So, the image of $z_3 = (xy + yx)^3$ in $\mathcal{A}_2$ (in fact, even in $\mathcal{A}_1$) is a central element.

Since $[y, yRy] \equiv y^2xy^2xy - yxy^2xy^2 \equiv [y^2xy^2x + xy^2xy^2, y] \bmod (y^3)$, then $[y, yRy + xy^2xy^2 + y^2xy^2x] \equiv 0 \bmod (y^3)$. Obviously $[x, yRy + xy^2xy^2 + y^2xy^2x] \equiv 0 \bmod (x^2, [x, yRy])$. Let $z_4 = (xy^2 + yxy + y^2x)^2$. Then $z_4 \equiv yRy + xy^2xy^2 + y^2xy^2x \bmod (x^2, y^3)$, and the image of z_4 is a central element in

$$\mathcal{A}_3 = \mathcal{A}_2/(z_3) \cong k\langle x, y\rangle/\left(x^2, y^3, (xy + yx)^3\right).$$

Let $\mathcal{A}_4 = \mathcal{A}_3/(z_4) \cong k\langle x, y\rangle/(x^2, y^3, (xy + yx)^3, (xy^2 + yxy + y^2x)^2)$. Let $z_5 = (yxy^2 - \theta y^2xy)x + x(yxy^2 - \theta y^2xy)$, where $\theta \in k$ is a primitive cubic root of unit. Then $[x, z_5] \equiv 0 \bmod (x^2)$ and $z_5 y \equiv \theta y z_5 \bmod (z_1, z_2, z_4)$. So, z_5 is a normal element in $\mathcal{A}_4$.

Let $\mathcal{A}_5 = \mathcal{A}/(z_1, \cdots, z_5)$. By using the k-basis of $\mathcal{A}$, it is easy to see that $\mathcal{A}_5$ is spanned by the elements of the form $y^{n_y} B^{n_B} C^{n_C} A^{n_A} x^{n_x}$ with $n_x \le 1, n_y \le 2, n_A \le 2, n_B \le 1, n_C = 0$, which implies that $\mathcal{A}_5$ is a finite-dimensional graded algebra.

It follows that $\mathcal{A}$ has enough normal elements. Hence $\mathcal{A}$ is Cohen–Macaulay and Auslander regular by [10, Theorem 0.2], and strongly Noetherian by [2, Proposition 4.9]. □

Definition 1.2. A connected graded algebra A is called rigid Gorenstein or satisfying the twisted Calabi–Yau condition if

$$\operatorname{Ext}^i_{A^e}\left(A, A^e\right) \cong \begin{cases} A_\nu, & i = d, \\ 0, & i \ne d. \end{cases}$$

for some integer d and graded automorphism ν of A, where ν is called the Nakayama automorphism of A, denoted by ν_A. If further, A is homologically smooth, then A is called twisted Calabi–Yau.

A twisted Calabi–Yau algebra A is Calabi–Yau if its Nakayama automorphism $\nu_A = \mathrm{id}$. Any Noetherian connected graded Artin–Schelter Gorenstein algebra satisfies the twisted Calabi–Yau condition [8].

Lemma 1.3. *(See [7], Lemma 1.5.) Let A be a Noetherian connected graded Artin–Schelter Gorenstein algebra and let z be a homogeneous ν_A-normal non-zero divisor of positive degree. Let σ be the element in* $\mathrm{Aut}(A)$ *such that $za = \sigma(a)z$ for all $a \in A$. Then $\nu_{A/(z)}$ is equal to $\nu_A \circ \sigma$ restricted to $A/(z)$.*

Proposition 1.4. *The extremal algebra $\mathcal{A}$ is a Calabi–Yau algebra.*

Proof. It suffices to show that the Nakayama automorphism of $\mathcal{A}$ is trivial. Since $\mathcal{A}$ can be viewed as a bigraded algebra, one can get that $\nu_{\mathcal{A}}(x) = rx$ and $\nu_{\mathcal{A}}(y) = sy$ for some $r, s \in k \setminus \{0\}$. Note that $\{z_1, z_2, z_3, z_4\}$ is a sequence of central elements and z_5 is a normal element. By Lemma 1.3, $\nu_{\mathcal{A}_4}(x) = rx$ and $\nu_{\mathcal{A}_4}(y) = sy$. Since $[x, z_5] = 0$, $z_5 y = \theta y z_5$ in $\mathcal{A}_5$, $\nu_{\mathcal{A}_5}(x) = rx$ and $\nu_{\mathcal{A}_5}(y) = -s\theta y$. Since $\mathcal{A}_5$ is a Frobenius algebra with a k-basis (it follows from diamond lemma [3]),

$$\left\{y^{n_y} B^{n_B} C^{n_C} A^{n_A} x^{n_x} \,\middle|\, n_x \le 1, n_y \le 2, n_A \le 2, n_B \le 1, n_C = 0\right\},$$

$$\nu_{\mathcal{A}_5}(x) = x \quad \text{and} \quad \nu_{\mathcal{A}_5}(y) = -\theta y.$$

It follows that $r = s = 1$, which implies that the Nakayama automorphism of $\mathcal{A}$ is trivial. Hence, the extremal algebra $\mathcal{A}$ is a Calabi–Yau algebra. □

Lemma 1.5. *Let $E = E_0 \oplus E_1 \oplus \cdots \oplus E_l$ be a connected graded Frobenius k-algebra and $\tau \in \mathrm{GrAut}(E)$. Then $\tau(u) = (\mathrm{hdet}\,\tau)^{-1} u$ for any $u \in E_l$, where $\mathrm{hdet}\,\tau$ is the homological determinant of τ.*

Proof. It follows from [6, Lemma 2.6]. $\square$

Theorem 1.6. *(See [7], Theorem 0.3.) Let A be a Noetherian connected graded Artin–Schelter regular algebra with Gorenstein parameter l. Then*

$$\nu_{A^\sigma} = \nu_A \circ \sigma^l \circ \xi^{-1}_{\mathrm{hdet}\,\sigma},$$

where A^σ is Zhang twist of A via $a * b = \sigma^{|b|}(a)b$, $\xi_{\mathrm{hdet}\,\sigma}(a) = (\mathrm{hdet}\,\sigma)^{|a|}a, \forall a \in A$, and $|a|$ is the degree of the homogeneous element a.

As remarked in [4, Remark 4.4], some deformed algebras of the extremal algebra are also Artin–Schelter regular algebras of dimension five. Zhou and Lu classified this class of algebras [11, Example 4.5].

Definition 1.7. The **generalized extremal algebra** $\mathcal{F}(p, q)$, where $p \neq 0, q \in k$ is the quotient algebra $k\langle x, y\rangle/(r_1, r_2, r_3)$ with relations

$$r_1 = yx^2 - p^2x^2y,$$
$$r_2 = y^3x - p^3xy^3,$$
$$r_3 = yRyx - p^4xyRy - p^8q\big(x^2yxy^3 - px^3y^4\big),$$

where $R \triangleq yxyx + pxy^2x + p^2xyxy$.

Remark 1. Proposition 1.1 holds for all the generalized extremal algebras.

It is easy to see that $\mathcal{F}(p, q)$ is the Zhang twist $\mathcal{F}(1, q)^\sigma$ as stated in [11, Example 4.5], where the automorphism σ is given by $\sigma(x) = p^{-1}x, \sigma(y) = y$. Notice that $\mathcal{A} = \mathcal{F}(1, 0)$.

Proposition 1.8. *(See [5], Proposition 2.4.) Let A be Noetherian Artin–Schelter Gorenstein and $\tau \in \mathrm{Aut}(A)$. If z is a normal non-zero divisor such that $\tau(z) = \lambda z$ for some $\lambda \in k \setminus \{0\}$, then $\mathrm{hdet}_A\,\tau = \lambda\,\mathrm{hdet}_{A/(z)}\,\tau$, where hdet is the homological determinant.*

Proposition 1.9. *The generalized extremal algebras $\mathcal{F}(p, q)$ are Artin–Schelter regular of dimension 5 with Gorenstein parameter $l = 12$. $\mathcal{F}(p, q)$ is Calabi–Yau if and only if $p = 1$.*

Proof. First we claim that the algebras $\mathcal{F}(1, q)$ are Calabi–Yau for any q. It is easy to check that $\{z_1, z_2, z_3, z_4, z_5\}$ is a normal sequence of elements in $\mathcal{F}(1, q)$ and x^2 is also central. Then by Lemma 1.3, $\nu_{\mathcal{F}(1,q)} = \nu_{\mathcal{A}}$ when restricted to the degree-one part. So the algebras $\mathcal{F}(1, q)$ are Calabi–Yau. Next, let us compute the Nakayama automorphism of $\mathcal{F}(p, q)$. By Theorem 1.6, $\nu_{\mathcal{F}(p,q)} = \sigma^{12} \circ \xi^{-1}_{\mathrm{hdet}\,\sigma}$. By using Proposition 1.8 and the same process as in Propositions 1.1,

$$\mathrm{hdet}\,\sigma = p^{-2}p^{-3}p^{-2}p^{-2}\,\mathrm{hdet}_{\mathcal{A}_5}\,\sigma = p^{-9}\,\mathrm{hdet}_{\mathcal{A}_5}\,\sigma.$$

Since $\mathcal{A}_5$ is a Frobenius algebra as noted in the proof of Proposition 1.4 and y^2xy^2xyxyx is a base element for $(\mathcal{A}_5)_{10}$ with $\sigma(y^2xy^2xyxyx) = p^{-4}y^2xy^2xyxyx$, $\mathrm{hdet}\,\sigma = p^{-9}\,\mathrm{hdet}_{\mathcal{A}_5}\,\sigma = p^{-5}$. So

$$\nu_{\mathcal{F}(p,q)}(x) = p^{-7}x, \qquad \nu_{\mathcal{F}(p,q)}(y) = p^5y.$$

Since $p^5 = p^7 = 1$, then $p = 1$. The conclusion follows. $\square$

2. Point modules over the extremal algebra

In this section, we describe the point modules over the extremal algebra. Let us first recall the definition of point modules.

Definition 2.1. (See [1, Definition 3.8].) Let A be a connected graded k-algebra generated in degree one. A graded A-module M is called a **point module** if M is cyclic and its Hilbert series $H_M(t) = (1 - t)^{-1}$.

Proposition 2.2. *All together, there are 6 families of isomorphism classes, which are all parameterized by k, and 3 isolated isomorphism classes of point modules over $\mathcal{A}$.*

Proof. Suppose that $P = \mathcal{A}e_0 = \bigoplus_{i=0}^{\infty} ke_i$ is a point module over $\mathcal{A}$ with $\deg e_i = i$. Then, for any $r \geq 0$, there are some $j_{r+1}, h_{r+1} \in k$ such that

$$xe_r = j_{r+1}e_{r+1}, \qquad ye_r = h_{r+1}e_{r+1}.$$

Obviously, for any fixed $r \in \mathbb{N}$, h_r and j_r are not all equal to zero. So, (j_r, h_r) can be viewed as a point in $\mathbb{P}^1$, and the point module P is determined uniquely by the infinite point sequence $(j_1, h_1), (j_2, h_2), \cdots, (j_r, h_r), \cdots$ up to isomorphism.

Now, we classify the point modules according to $x^2P \neq 0$ or $x^2P = 0$.

Case 1. Suppose that $x^2P \neq 0$. Since $x^2 \in \mathcal{A}$ is central and $ke_{i+2} = \mathcal{A}_{i+2}e_0 = \mathcal{A}_i e_2 \neq 0$, then $x^2e_0 \neq 0$, and $x^2P = \bigoplus_{i=2}^{\infty} ke_i$. Hence $xe_i \neq 0$ for all $i \geq 0$. After changing the k-basis of P if necessary, we may assume that $j_{r+1} = 1$ for all $r \geq 0$.

It follows from the $\mathcal{A}$-module structure of P and the first two relations $[x^2, y]$ and $[x, y^3]$ in $\mathcal{A}$ that

$$\begin{cases} h_{r+1} = h_{r+3}, \\ h_{r+1}h_{r+2}h_{r+3} = h_{r+2}h_{r+3}h_{r+4}, \end{cases} \quad (\forall r \geq 0). \tag{2.1}$$

This is equivalent to the following system of equations

$$\begin{cases} h_{r+1} = h_{r+3}, \\ h_{r+1}h_{r+2}(h_{r+1} - h_{r+2}) = 0, \end{cases} \quad (\forall r \geq 0). \tag{2.2}$$

Then it is easy to see that (2.2) has the following three solutions:

- for any $r \in \mathbb{N}$, $h_1 = h_r$;
- for any $r \in \mathbb{N}$, $h_{2r+1} = 0$ and $h_2 = h_{2r} \neq 0$;
- for any $r \in \mathbb{N}$, $h_{2r} = 0$ and $h_1 = h_{2r+1} \neq 0$.

The third relation $[x, yRy]$ in $\mathcal{A}$ induces the equations

$$\begin{aligned} &h_{r+1}h_{r+3}h_{r+5}h_{r+6} + h_{r+1}h_{r+3}h_{r+4}h_{r+6} + h_{r+1}h_{r+2}h_{r+4}h_{r+6} \\ &\quad = h_{r+2}h_{r+4}h_{r+6}h_{r+7} + h_{r+2}h_{r+4}h_{r+5}h_{r+7} + h_{r+2}h_{r+3}h_{r+5}h_{r+7} \quad (r \geq 0), \end{aligned}$$

which are satisfied by all the solutions to (2.2).

Each solution above gives one family of isomorphism class of the point modules. The corresponding infinite point sequences in $\mathbb{P}^1$ are

(I) $(1, h), (1, h), (1, h), \cdots (1, h), \cdots,$
(II) $(1, 0), (1, h), (1, 0), (1, h), \cdots, (1, 0), (1, h), \cdots,$
(III) $(1, h), (1, 0), (1, h), (1, 0), \cdots, (1, h), (1, 0), \cdots.$

So, in this case, we have three families of isomorphism class of the point modules over $\mathcal{A}$, which are parameterized by k, k^* and k^*, respectively.

Case 2. Suppose $x^2P = 0$. Let $\mathcal{A}_1 = \mathcal{A}/(x^2)$. Then P can be viewed as a point module over $\mathcal{A}_1$. So it turns out to figure out the point modules over $\mathcal{A}_1$. It follows from the module structure of P and the generating relations of $\mathcal{A}$ that, for all $r \geq 0$,

$$\begin{cases} j_{r+1}j_{r+2} = 0, \\ h_{r+1}h_{r+2}h_{r+3}j_{r+4} = j_{r+1}h_{r+2}h_{r+3}h_{r+4}, \\ h_{r+1}j_{r+2}h_{r+3}j_{r+4}h_{r+5}h_{r+6}j_{r+7} + h_{r+1}j_{r+2}h_{r+3}h_{r+4}j_{r+5}h_{r+6}j_{r+7} + h_{r+1}h_{r+2}j_{r+3}h_{r+4}j_{r+5}h_{r+6}j_{r+7} \\ \quad = j_{r+1}h_{r+2}j_{r+3}h_{r+4}j_{r+5}h_{r+6}h_{r+7} + j_{r+1}h_{r+2}j_{r+3}h_{r+4}h_{r+5}j_{r+6}h_{r+7} + j_{r+1}h_{r+2}h_{r+3}j_{r+4}h_{r+5}j_{r+6}h_{r+7}. \end{cases} \tag{2.3}$$

Multiplying the third equation by $j_{r+2}j_{r+4}$, we get $h_{r+1}j_{r+2}^2h_{r+3}j_{r+4}^2h_{r+5}h_{r+6}j_{r+7} = 0$ and so $h_{r+1}j_{r+2}h_{r+3}j_{r+4}h_{r+5}h_{r+6}j_{r+7} = 0$. Then it is easy to see that the system of Eqs. (2.3) is equivalent to that

$$\begin{cases} j_{r+1}j_{r+2} = 0, \\ h_{r+1}h_{r+2}h_{r+3}j_{r+4} = j_{r+1}h_{r+2}h_{r+3}h_{r+4}, \\ h_{r+1}j_{r+2}h_{r+3}j_{r+4}h_{r+5}h_{r+6}j_{r+7} = 0, \\ h_{r+1}j_{r+2}h_{r+3}h_{r+4}j_{r+5}h_{r+6}j_{r+7} = 0, \\ h_{r+1}h_{r+2}j_{r+3}h_{r+4}j_{r+5}h_{r+6}j_{r+7} = 0, \\ j_{r+1}h_{r+2}j_{r+3}h_{r+4}j_{r+5}h_{r+6}h_{r+7} = 0, \\ j_{r+1}h_{r+2}j_{r+3}h_{r+4}h_{r+5}j_{r+6}h_{r+7} = 0, \\ j_{r+1}h_{r+2}h_{r+3}j_{r+4}h_{r+5}j_{r+6}h_{r+7} = 0, \end{cases} \quad (\forall r \geq 0). \tag{2.4}$$

The infinite point sequence $\{(j_r, h_r) \mid r \geq 1\}$ is a sequence associated with some point module P if and only if it is a solution to the system of Eqs. (2.4).

Furthermore, if we define a new sequence $\{(u_r, v_r) \mid r \in \mathbb{N}\}$ by $u_r := j_{r+1}$; $v_r := h_{r+1}$, $\forall r \in \mathbb{N}$, then the sequence $\{(u_r, v_r) \mid r \in \mathbb{N}\}$ is also a solution to (2.4).

Now let us solve the following first.

$$\begin{cases} j_{r+1} j_{r+2} = 0, \quad 0 \le r \le 5, \\ h_{r+1}h_{r+2}h_{r+3} j_{r+4} = j_{r+1}h_{r+2}h_{r+3}h_{r+4}, \quad 0 \le r \le 3, \\ h_1 j_2 h_3 j_4 h_5 h_6 j_7 = 0, \\ h_1 j_2 h_3 h_4 j_5 h_6 j_7 = 0, \\ h_1 h_2 j_3 h_4 j_5 h_6 j_7 = 0, \\ j_1 h_2 j_3 h_4 j_5 h_6 h_7 = 0, \\ j_1 h_2 j_3 h_4 h_5 j_6 h_7 = 0, \\ j_1 h_2 h_3 j_4 h_5 j_6 h_7 = 0. \end{cases} \tag{2.5}$$

To find the isomorphism class, we may assume that $h_r = 1$ (resp. $j_r = 1$) if $j_r = 0$ (resp. $h_r = 0$). Then the system of Eqs. (2.5) has the following 7 solutions.

- If $j_1 = j_2 = 0$, then the solutions are

$$(1) \quad \left(\begin{array}{c|c|c|c|c|c|c} j_1=0 & j_2=0 & j_3=0 & j_4=0 & j_5=0 & j_6=0 & j_7=0 \\ h_1=1 & h_2=1 & h_3=1 & h_4=1 & h_5=1 & h_6=1 & h_7=1 \end{array}\right),$$

$$(2) \quad \left(\begin{array}{c|c|c|c|c|c|c} j_1=0 & j_2=0 & j_3=1 & j_4=0 & j_5=0 & j_6=1 & j_7=0 \\ h_1=1 & h_2=1 & h_3 & h_4=1 & h_5=1 & h_6=h_3 & h_7=1 \end{array}\right),$$

$$(3) \quad \left(\begin{array}{c|c|c|c|c|c|c} j_1=0 & j_2=0 & j_3=1 & j_4=0 & j_5=1 & j_6=0 & j_7=0 \\ h_1=1 & h_2=1 & h_3=0 & h_4=1 & h_5=0 & h_6=1 & h_7=1 \end{array}\right).$$

- If $j_1 = 0$, $j_2 \neq 0$, then the solutions are

$$(4) \quad \left(\begin{array}{c|c|c|c|c|c|c} j_1=0 & j_2=1 & j_3=0 & j_4=0 & j_5=1 & j_6=0 & j_7=0 \\ h_1=1 & h_2 & h_3=1 & h_4=1 & h_5=h_2 & h_6=1 & h_7=1 \end{array}\right),$$

$$(5) \quad \left(\begin{array}{c|c|c|c|c|c|c} j_1=0 & j_2=1 & j_3=0 & j_4=1 & j_5=0 & j_6=1 & j_7=0 \\ h_1=1 & h_2=0 & h_3=1 & h_4=0 & h_5=1 & h_6=0 & h_7=1 \end{array}\right).$$

- If $j_1 \neq 0$, $j_2 = 0$, then the solutions are

$$(6) \quad \left(\begin{array}{c|c|c|c|c|c|c} j_1=1 & j_2=0 & j_3=0 & j_4=1 & j_5=0 & j_6=0 & j_7=1 \\ h_1 & h_2=1 & h_3=1 & h_4=h_1 & h_5=1 & h_6=1 & h_7=h_1 \end{array}\right),$$

$$(7) \quad \left(\begin{array}{c|c|c|c|c|c|c} j_1=1 & j_2=0 & j_3=1 & j_4=0 & j_5=1 & j_6=0 & j_7=1 \\ h_1=0 & h_2=1 & h_3=0 & h_4=1 & h_5=0 & h_6=1 & h_7=0 \end{array}\right).$$

It follows that the solutions to (2.4) are

- (1)(1)(1) $\cdots$,
- (2)(4)(6) (2)(4)(6)(2)(4)(6) $\cdots$ (2)(4)(6) $\cdots$,
- (4)(6)(2) (4)(6)(2)(4)(6)(2) $\cdots$ (4)(6)(2) $\cdots$,
- (6)(2)(4) (6)(2)(4)(6)(2)(4) $\cdots$ (6)(2)(4) $\cdots$,
- (5)(7) (5)(7)(5)(7) $\cdots$ (5)(7) $\cdots$,
- (7)(5) (7)(5)(7)(5) $\cdots$ (7)(5) $\cdots$.

The corresponding infinite point sequences in $\mathbb{P}^1$ are

(IV) $(0,1), (0,1), (0,1), \cdots (0,1), \cdots$,
(V) $(0,1), (0,1), (1,h), (0,1), (0,1), (1,h), \cdots, (0,1), (0,1), (1,h), \cdots$,
(VI) $(0,1), (1,h), (0,1), (0,1), (1,h), (0,1), \cdots, (0,1), (1,h), (0,1), \cdots$,
(VII) $(1,h), (0,1), (0,1), (1,h), (0,1), (0,1), \cdots, (1,h), (0,1), (0,1), \cdots$,
(VIII) $(0,1), (1,0), (0,1), (1,0), \cdots, (0,1), (1,0), \cdots$,
(IX) $(1,0), (0,1), (1,0), (0,1), \cdots, (1,0), (0,1), \cdots$.

So, in this case, we have three families of isomorphism class of the point modules over $\mathcal{A}$ given by (V), (VI) and (VII), which are all parameterized by k, and three isolated class given by (IV), (VIII) and (IX).

To sum up, up to isomorphisms, there are 6 families of isomorphism classes, which are all parameterized by k, and 3 isolated isomorphism classes of point modules over $\mathcal{A}$. $\square$

To determine the isomorphism classes of point modules over the generalized extremal algebras $\mathcal{F}(p,q)$, we prove an easy fact that the twisted equivalence [9] preserves the point modules.

Lemma 2.3. *Suppose that the connected graded algebras A and B are twisted equivalent. Then the equivalence preserves the point modules.*

Proof. *Suppose $B \cong A^\tau$ for some twisting system τ of A. Then, GrMod $A \cong \mathrm{GrMod} A^\tau$ and the equivalence functor is defined by sending an A-module M to the A^τ-module M^τ, which is equal to M as vector spaces. The conclusion follows from the fact that the functor preserves the cyclic modules.* $\square$

Corollary 2.4. *All together, there are 6 families of isomorphism classes, which are all parameterized by k, and 3 isolated isomorphism classes of point modules over the generalized extremal algebras $\mathcal{F}(p,q)$.*

Proof. By Lemma 2.3, it suffices to show the result for the algebras $\mathcal{F}(1,q)$. The proof follows from that of Proposition 2.2, which is independent of q. $\square$

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