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Differential geometry

Heat equation in a model matrix geometry ☆

L'équation de la chaleur pour une géométrie matricielle modèle

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ABSTRACT

In this paper, we study the heat equation in a model matrix geometry M_n. Our main results are about the global behavior of the heat equation on M_n. We can show that if c_0 is the initial positive definite matrix in M_n, then $c(t)$ exists for all time and is positive definite too. We can also show the entropy stability of the solutions to the heat equation.

RÉSUMÉ

Dans cet article, nous étudions l'équation de la chaleur pour la géométrie matricielle modèle M_n. Nos principaux résultats concernent le comportement global de l'équation de la chaleur. Nous parvenons à montrer que, si la matrice initiale c_0 est définie positive dans M_n, alors $c(t)$ existe pour tout temps et reste définie positive. Nous montrons également la stabilité de l'entropie des solutions de l'équation de la chaleur.

1. Introduction

In Riemannian geometry, the spectrum of the Laplacian operator on a compact Riemannian manifold gives geometric and topological information about the manifold. The heat equation proof of the Atiyah–Singer index theorem is one of the most famous examples [22]. In particular, through the use of the heat equation, one can define the curvature of the compact n-dimensional Riemannian manifold (M, g) as below. Let $H(x, y, t)$ be the heat kernel of the Laplacian operator [21]. Let (λ_j) be the spectrum and $\{\phi_j(x)\}$ the corresponding eigenfunctions on M. Then the heat kernel $H(x, y, t)$ can be written as

$$H(x, y, t) = \sum_j \mathrm{e}^{-\lambda_j t}\phi_j(x)\phi_j(y).$$

Then we have the expansion

$$(4\pi t)^{n/2} H(x, x, t) = 1 + \frac{t}{3}R + 0\big(t^2\big)$$

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near $t = 0$. Here R is the scalar curvature of the metric g. This implies that we can define the scalar curvature by the formula:

$$R = 3\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}\left[(4\pi t)^{n/2}H(x,x,t)\right].$$

Hence, it is natural to use the heat equation to define the scalar curvature in the non-commutative geometry [2–6,10,17,18]. The aim of this paper is to explore the properties of heat equation in a simple case, which has been recently studied by R. Duvenhage in [8]. In [8], the author has introduced the Ricci flow and his main results can be briefly stated as follows. Let M_n be the C^* algebra generated by the two matrices [19]

$$u = \mathrm{e}^{2\pi\,\mathrm{i}x/n}, \qquad v = \mathrm{e}^{2\pi\,\mathrm{i}y/n},$$

where x, y are two Hermitian matrices on C^n. Define the derivations on M_n

$$\delta_1 := [y,\cdot], \qquad \delta_2 := -[x,\cdot]$$

and the Laplacian operator

$$\Delta = \delta_1^2 + \delta_2^2.$$

Then the Ricci flow can be defined by [8]:

$$\frac{\mathrm{d}}{\mathrm{d}t}c(t) = -\Delta\log c(t).$$

For any positive definite matrix $c_0 \in M_n$, there is a global solution $c(t)$ to the Ricci flow for all positive t, and the solution converges to the scalar matrix determined by the initial data c_0. Furthermore, along this flow, the von Neumann entropy of the solution $c(t)$ is increasing, except c_0, which is a scalar matrix.

We can introduce the eigenvalues and eigenfunctions of the Laplacian operator Δ in the same M_n and define the heat kernel and the scalar curvature as above. Then we can introduce the Ricci flow in the standard way as

$$\frac{\mathrm{d}}{\mathrm{d}t}c(t) = R\big(c(t)\big)c(t)$$

for the positive definite matrix $c(t)$. This gives the fourth way to define the Ricci flow in non-commutative geometry. However, because there is no explicit relation about the scalar curvature and the matrix $c(t)$, this approach may be very complicated for us to get a global Ricci flow as in [13,14]. The Ricci flow has been used to find many interesting applications in physics. It appears as the renormalization group equations of 2-dimensional sigma models [11,12]. It can be used to study the evolution of the ADM mass in asymptotically flat spaces [7]. More recently, it appears in the study of the contribution of black holes in Euclidean quantum gravity [16,15]. In [1], the author describes an appropriate analog of Hamilton's Ricci flow for the noncommutative two tori, which is the prototype example of noncommutative manifolds. It is still an interesting question to find more ways to define the Ricci flow in noncommutative geometry.

Our main results are about the global behavior of the heat equation on M_n. We can show that if c_0 is the initial positive definite matrix in M_n, then the solution $c(t)$ exists for all $t \geq 0$ and is positive definite too. We can also show the entropy stability of the solutions of the heat equation. The plan of the paper is below. In Section 2, we recall some properties of the noncommutative geometry model M_n. In Section 3, we study the behavior of the solution of the heat equation on M_n. We consider the entropy stability of the solutions of the heat equation in Section 4.

2. Elementary noncommutative differential geometry

Let x, y be two Hermitian matrices on C^n. Define $u = \mathrm{e}^{\frac{2\pi\mathrm{i}}{n}x}$, $v = \mathrm{e}^{\frac{2\pi\mathrm{i}}{n}y}$. We use M_n to denote the algebra of all $n \times n$ complex matrices generated by U and V with the bracket $[u,v] = uv - vu$. Then CI, which is the scalar multiple of the identity matrices I, is the commutator of the operation $[u,v]$. Sometimes we simply use 1 to denote the $n \times n$ identity matrix.

We define two derivations δ_1 and δ_2 on the algebra M_n by the commutators

$$\delta_1 := [y,\cdot], \qquad \delta_2 := -[x,\cdot].$$

Define the Laplacian operator on M_n by

$$\Delta = \delta_1^2 + \delta_2^2 = \delta_\mu\delta_\mu,$$

where we have used the Einstein sum convention. We use the Hilbert–Schmidt norm defined by the inner product

$$\langle a, b\rangle = \tau\left(a^*b\right)$$

on the algebra M_n. Here a^* is the Hermitian adjoint of the matrix a and τ denotes the usual trace function on M_n. We now state basic properties of δ_1, δ_2 and Δ [8,19] as follows.

Proposition 2.1. *There exists a positive constant c such that for any $a \in M_n$,*

$$c|a-\bar{a}|^2 \le \langle \delta_\mu(a-\bar{a}), \delta_\mu(a-\bar{a})\rangle \le c^{-1}|a-\bar{a}|^2,$$

where $\bar{a} = \frac{\tau(a)}{n} I$ and $\tau(a)$ denotes the trace of a.

Proof. Define

$$|a-\bar{a}|_1 = \langle \delta_\mu(a-\bar{a}), \delta_\mu(a-\bar{a})\rangle^{\frac{1}{2}}$$

and we can verify that $|\cdot|_1$ is a norm on M_n/CI.

We need to verify the following three conditions.

(1) For $\bar{a}=0$, if $|a|_1 = 0 \Leftrightarrow \delta_\mu(a) = 0$, $\mu = 1, 2$. Then by [8], we know that $a \in CI$. So $a = \bar{a} = 0$.
(2) $\forall \lambda > 0$, $|\lambda a|_1 = |\lambda||a|_1$ is clearly true.
(3) $\forall a, b \in M_n/CI$, it is also true that

$$|a+b|_1 \le |a|_1 + |b|_1.$$

Since M_n/CI is a finite-dimension vector space, the Hilbert–Schmidt norm $|\cdot|$ is equivalent to $|\cdot|_1$ on M_n/CI. □

The following result is basically proved in Proposition 3.1 in [8] and we refer to [8] for the proof.

Proposition 2.2. *For any positive definite matrix $a \in M_n$, $\forall m \in Z$, we have*

$$\tau\left(a^m \Delta a\right) \ge 0$$

with equality if and only if $a \in CI$, i.e. if and only if a is a scalar multiple of the identity matrix I.

3. Heat equation

In this section, we study the heat equation

$$u_t = -\Delta u, \quad u \in M_n, \tag{3.1}$$

with the initial data $u|_{t=0} = u_0$.

The heat equation (3.1) can be considered as an ODE on M_n, so it has a local unique solution $u = u(t)$ for small $t > 0$.

We now study the global property of the solution $u(t)$. Note that $\bar{u}_t = \frac{1}{n}\frac{\mathrm{d}}{\mathrm{d}t}\tau(u) = \frac{1}{n}\tau(u_t) = -\frac{1}{n}\tau(\Delta u) = 0$. Applying Proposition 2.1, we obtain:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|u-\bar{u}|^2 &= \langle u-\bar{u}, u_t\rangle + \langle u_t, u-\bar{u}\rangle \\ &= -\langle u-\bar{u}, \Delta u\rangle - \langle \Delta u, u-\bar{u}\rangle \\ &= -2\langle \delta_\mu u, \delta_\mu u\rangle \\ &\le -2c|u-\bar{u}|^2.\end{aligned}$$

Note that

$$|u-\bar{u}|^2(t) \le |u-\bar{u}|^2(0)\,\mathrm{e}^{-2ct} \to 0 \tag{3.2}$$

as $t \to \infty$. The heat equation (3.1) has a unique global solution for $t \ge 0$ and $\bar{u} = \lim_{t\to\infty} u(t) = \bar{u}_0$. We see from (3.2) that if the initial data $u(0)$ is Hermitian, then the solution $u(t)$ is also Hermitian. In fact, let $u(t)$ and $v(t)$ be two solutions corresponding to the initial datum $u(0)$ and $u^*(0)$, respectively. Let $W(t) = u(t) - v(t)$. Then $W(0) = 0$. By (3.2), we have $W(t) = 0$ for $t > 0$. Note that by Proposition 2.1 in [8], we know that

$$u_t^* = (u_t)^* = -(\Delta u)^* = -(\delta_\mu \delta_\mu u)^* = \delta_\mu(\delta_\mu u)^* = -\delta_\mu\delta_\mu u^* = -\Delta u^*.$$

By (3.2), we have $v(t) = u^*(t)$. Then $u(t) = v(t) = u^*(t)$ for $t > 0$, which implies that $u(t)$ is Hermitian.

Assume $u_0 > 0$, we claim that $u(t) > 0$. Here is the proof. We compute:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\log\det u &= \tau\left(u^{-1}u_t\right) \\ &= -\tau\left(u^{-1}\Delta u\right) \\ &= -\left\langle u^{-1}, \Delta u\right\rangle \\ &= -\left\langle \delta_\mu u^{-1}, \delta_\mu u\right\rangle.\end{aligned}$$

Note that

$$\delta_1 u^{-1} = \left[y, u^{-1}\right] = y u^{-1} - u^{-1} y = u^{-1}(uy - yu)u^{-1} = -u^{-1}\delta_1 u u^{-1}.$$

Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\log\det u &= \left\langle u^{-1}\delta_\mu u u^{-1}, \delta_\mu u\right\rangle \\ &= \tau\left(u^{-1}(\delta_\mu u)^* u^{-1}\delta_\mu u\right) \\ &= \tau\left(u^{-1/2}(\delta_\mu u)^* u^{-1/2} u^{-1/2}\delta_\mu u u^{-1/2}\right) \\ &= \left\langle u^{-1/2}(\delta_\mu u) u^{-1/2}, u^{-1/2}\delta_\mu u u^{-1/2}\right\rangle \\ &> 0.\end{aligned}$$

That is to say, $\log\det u$ is an increasing function in t. Hence $\det u > 0$ for $t > 0$. So $u(t) > 0$ for $t > 0$. This completes the proof of the claim.

In conclusion, we have proven the following result.

Theorem 3.1. *For any $u_0 \in M_n$, the heat equation (3.1) has a global solution $u(t)$ for $t \ge 0$ with its limit $\bar{u}_0$ as $t \to \infty$. Furthermore, (1) if $u_0 > 0$, then $u(t) > 0, \forall t > 0$; (2) if the initial data $u(0)$ is Hermitian, then the solution $u(t)$ is also Hermitian.*

In below, we assume $u_0 > 0$ and define the von Neumann entropy by

$$S(u) = -\tau(u\log u)$$

for the positive solution $u = u(t)$ with $u(0) = u_0$.

We have the following result.

Proposition 3.2. *The entropy $S(u)$ is increasing along the heat equation (3.1).*

Proof. According to the proof of Theorem 4.1 in [8], we have:

$$\frac{d}{dt}S(u) = -\tau(u_t\log u) = \tau(\Delta u\log u) = \tau(u\Delta\log u).$$

Set $l = \log u$. Then $u = \mathrm{e}^l$.

By Proposition 2.2, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}S(u) = \tau\left(\mathrm{e}^l\Delta l\right) \ge 0.$$

So $S(u)$ is increasing along the heat equation (3.1). □

4. Entropy stability of the heat equation

Given two initial matrices $u(0)$, $v(0)$ in the unit ball of M_n. Let u, v be the corresponding solutions of the heat equation with the initial datum $u(0)$ and $v(0)$ respectively. Then the solutions u and v are also in the unit ball of M_n. Let $w = \bar{u}(0) - \bar{v}(0)$. By Proposition 2.1, we obtain:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|u - v - w|^2 &= -\langle u - v - w, \Delta u - \Delta v\rangle - \langle\Delta u - \Delta v, u - v - w\rangle \\ &= -2\left\langle\delta_\mu(u - v - w), \delta_\mu(u - v - w)\right\rangle \\ &\le -2c|u - v - w|^2.\end{aligned}$$

Then

$$|u - v - w|^2(t) \le A\,\mathrm{e}^{-2ct} \to 0,$$

where $A = |u - v - w|^2(0)$.

Note that $|w| \le |u(0) - v(0)|$. By the triangle inequality and the result above, we know that

$$|u - v|(t) \le |u - v - w|(t) + |w| \le C|u - v|(0) \tag{4.1}$$

for some uniform constant $C \ge 1$. This is the Hilbert–Schmidt norm stability of Eq. (3.1).

Remark. Recall here the definition of the trace norm $T(a,b)$ [20] for $a,b \in M_n$. By the spectral decomposition theorem, we decompose $a-b=Q-P$, where Q and R are positive operators with orthogonal support. We define $T(a,b)=\mathrm{tr}(Q)+\mathrm{tr}(P)$, which is independent of the choice of Q and R [20]. Clearly, we have $T(a,b)\le\sqrt{n}|a-b|$. Note also that $|a-b|\le T(a,b)$. Using the stability result (4.1), we have the trace norm stability of solutions $u(t)$ and $v(t)$, namely, there is a uniform constant $C_1>0$ such that for any $t>0$, it holds:

$$T(u,v)(t)\le C_1T(u,v)(0). \tag{4.2}$$

We now recall the Fannes inequality [9] (see Theorem 11.6 in [20]). The Fannes inequality states that for any $a,b\in M_n$ with $a>0, b>0$ and $T(a,b)\le\frac{1}{e}$, we have

$$\big|S(a)-S(b)\big|\le T(a,b)\log n+\eta\big(T(a,b)\big), \tag{4.3}$$

where $\eta(s)=-s\log s$, n is the dimension of M_n. Then we can use the Fannes inequality to get the entropy stability of the solution of (3.1).

Theorem 4.1. *Let $u(0)$ and $v(0)$ be as above. If $T(u(0),v(0))\le\frac{1}{C_1e}$, $u(0)>0$, $v(0)>0$ in M_n, then the solutions $u(t)$, $v(t)$ (with initial datum $u(0)$, $v(0)$ respectively) satisfy*

$$\big|S\big(u(t)\big)-S\big(v(t)\big)\big|\le C_1T(u,v)(0)\log n+\eta\big(C_1\big(T(u,v)\big)(0)\big),$$

where C_1 is the uniform constant in (4.2).

Proof. By the Fannes inequality (4.3) and the trace norm stability inequality (4.2), we have

$$\begin{aligned}\big|S\big(u(t)\big)-S\big(v(t)\big)\big| &\le T\big(u(t),v(t)\big)\log n+\eta\big(T\big(u(t),v(t)\big)\big)\\ &\le C_1T\big(u(0),v(0)\big)\log n+\eta\big(C_1T\big(u(0),v(0)\big)\big),\end{aligned}$$

where we have used the monotonicity of the function η in $[0,\frac{1}{e}]$. This completes the proof. □

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