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Complex analysis

On the regularization of J-plurisubharmonic functions

Sur la régularisation des fonctions J-pluri-sous-harmoniques

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A R T I C L E I N F O

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A B S T R A C T

We show that on almost complex surfaces plurisubharmonic functions can be locally approximated by smooth plurisubharmonic functions. The main tool is the Poletsky type theorem due to U. Kuzman.

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R É S U M É

Nous montrons que, sur une surface presque complexe, les fonctions pluri-sous-harmoniques peuvent être localement approximées par des fonctions pluri-sous-harmoniques lisses. La méthode consiste à appliquer le théorème de type Polestsky démontré par U. Kuzman.

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1. Introduction

Let (M, J) be an almost complex manifold. In his paper [3], Haggui defines plurisubharmonic functions on M as upper semicontinuous functions that are subharmonic on every J-holomorphic disk. Recently Harvey and Lawson proved that a locally integrable function u is plurisubharmonic iff a current $\mathrm{i}\partial\bar{\partial}u$ is positive (see [4]).

It is a very natural open question in this theory whether any plurisubharmonic function is (locally) a limit of a decreasing sequence of smooth plurisubharmonic functions. The Richberg-type theorem was proved in [8]. This gives a positive answer in a case of continuous functions. In this note, we prove it for all plurisubharmonic functions in the (complex) dimension[2] 2.

Theorem 1. *Let $\dim M = 2$ and $P \in M$. Then there is a domain D, which is a neighborhood of P, such that for every $u \in \mathcal{PSH}(D)$ there exists a decreasing sequence $\psi_k \in \mathcal{C}^\infty \cap \mathcal{PSH}(D)$ such that $\psi_k \to u$.*

As an immediate consequence of Theorem 1 and Proposition 5.2 from [8], we obtain the following:

Corollary 2. *Let $\dim M = 2$ and u, v in $W^{1,2}_{loc} \cap \mathcal{PSH}(M)$. Then a current $\mathrm{i}\partial\bar{\partial}u \wedge \mathrm{i}\partial\bar{\partial}v$ defined in [8] is a (positive) measure.*

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[2] In this note by the dimension of an almost complex manifold we mean the complex dimension which is a half of the real dimension.

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In particular the Monge–Ampère operator $(\mathrm{i}\partial\bar{\partial}u)^2$ is well defined for any bounded plurisubharonic function u on an almost complex surface (compare to Proposition 4.2 in [8]). On domains in $\mathbb{C}^2$ it was proved by Błocki (see [1]) that a set $W^{1,2}_{\mathrm{loc}} \cap \mathcal{PSH}$ is a natural domain for the Monge–Ampère operator.

The main step in the proof of Theorem 1 is the continuity of largest plurisubharmonic minorants of certain continuous functions. Harvey and Lawson, after viewing a preliminary version of this paper, informed the author that using viscosity methods it is possible to prove it in any dimension. It will be explained in [5].

2. Proof

2.1. J-holomorphic discs

A good reference for the (local) theory of J-holomorphic discs is [6]. In this subsection, J is $\mathcal{C}^1$ close to J_{st} (in particular $(J + J_{\mathrm{st}})$ is invertible), where J_{st} is the standard (integrable) almost complex structure in $\mathbb{C}^n$. Let $\mathbb{D}$ be a unit disc in $\mathbb{C}$. A function $u : \mathbb{D} \to (\mathbb{C}^n, J)$ is J-holomorphic if and only if

$$\frac{\partial u}{\partial \bar{z}} + Q(u)\frac{\partial u}{\partial z} = 0$$

when

$$Q = (J + J_{\mathrm{st}})^{-1}(J - J_{\mathrm{st}}).$$

Let $0 < \alpha < 1$ and $T : \mathcal{C}^{0,\alpha}(\bar{\mathbb{D}}, \mathbb{C}^n) \to \mathcal{C}^{1,\alpha}(\bar{\mathbb{D}}, \mathbb{C}^n)$ be the Cauchy–Green operator given by:

$$Tu = \frac{1}{\pi}\int_{\mathbb{D}} \frac{u(\zeta)}{\cdot - \zeta}\mathrm{d}\zeta.$$

Recall that $\bar{\partial}(Tu) = u$ for $u \in \mathcal{C}^{0,\alpha}(\bar{\mathbb{D}}, \mathbb{C}^n)$. Set

$$\Phi u = u + T\left(Q(u)\frac{\partial u}{\partial z}\right)$$

and

$$\Psi u = \Phi u + (u - \Phi u)(0).$$

By the definition $\Psi u(0) = u(0)$. Note that $u \in \mathcal{C}^{1,\alpha}(\bar{\mathbb{D}}, \mathbb{C}^n)$ is J-holomorphic in $\mathbb{D}$ iff Ψu is J_{st}-holomorphic. Because $\mathrm{d}\Psi$ is close to Id, the map $\Psi : \mathcal{C}^{1,\alpha}(\bar{\mathbb{D}}, \mathbb{C}^n) \to \mathcal{C}^{1,\alpha}(\bar{\mathbb{D}}, \mathbb{C}^n)$ is a local diffeomorphism and there is a constant C_0 such that $\|(\mathrm{d}\Psi)^{-1}\| \le C_0$ everywhere.

We will use the following lemma.

Lemma 3. *Let $V \in \mathbb{C}^n$. For any $u \in \mathcal{C}^{1,\alpha}(\bar{\mathbb{D}}, \mathbb{C}^n)$, there is $v \in \mathcal{C}^{1,\alpha}(\bar{\mathbb{D}}, \mathbb{C}^n)$ such that $\Psi v = \Psi u + V$ and $\|u - v\|_{\mathcal{C}^{1,\alpha}} \le C_0|V|$.*

Proof. Set $U_t = \Psi u + tV$ and

$$S = \left\{t \in [0,1] : \exists w \in \mathcal{C}^{1,\alpha}\left(\bar{\mathbb{D}}, \mathbb{C}^n\right) \text{ s. t. } \Psi w = U_t,\ \|u - w\|_{\mathcal{C}^{1,\alpha}} \le tC_0|V|\right\}.$$

S is nonempty, by the inverse function theorem it is open, by the Arzelà–Ascoli theorem it is closed and hence $S = [0, 1]$.

2.2. Disc envelope

Let $p \in \Omega \subset M$ and let $\mathcal{O}_p(\bar{\mathbb{D}}, \Omega)$ be a set of J-holomorphic discs $\lambda : \bar{\mathbb{D}} \to \Omega$ with $\lambda(0) = p$. For an upper semicontinuous function $f : \Omega \to \mathbb{R}$, we consider the following disc envelope:

$$P_\Omega f(p) = \inf_{\lambda \in \mathcal{O}_p(\bar{\mathbb{D}}, \Omega)} \frac{1}{2\pi}\int_0^{2\pi} f \circ \lambda\left(\mathrm{e}^{\mathrm{i}t}\right)\mathrm{d}t.$$

We need the following lemma.

Lemma 4. *Let $\Omega_1 \Subset \Omega_2 \subset \mathbb{C}^n$ and let J be an almost complex structure on Ω_2, which is $\mathcal{C}^1$ close to J_{st}. Let $f \in \mathcal{C}(\Omega_2)$ be such that*

$$P_{\Omega_1} f = (P_{\Omega_2} f)|_{\Omega_1}.$$

Then $P_{\Omega_1} f \in \mathcal{C}(\Omega_1)$.

Proof. By shrinking Ω_2, we can assume that f is uniformly continuous on Ω_2 with a modulus of continuity ω and J is $\mathcal{C}^1$ close to J_{st} on $\mathbb{C}^n$. Set any $0 < \delta < C_0^{-1}\operatorname{dist}(\partial\Omega_1, \partial\Omega_2)$. Let $\varepsilon > 0$, and $p, q \in \Omega_1$ with $|p - q| \le \delta$. There is $\lambda \in \mathcal{O}_p(\bar{\mathbb{D}}, \Omega_1)$ such that:

$$P_{\Omega_1} f(p) \ge \frac{1}{2\pi}\int_0^{2\pi} f \circ \lambda\big(e^{it}\big)dt - \varepsilon.$$

By Lemma 3, there is a function $\mu \in \mathcal{C}^{1,\alpha}(\bar{\mathbb{D}}, \mathbb{C}^n)$ such that:

$$\Psi(\mu) = \Psi(\lambda) + p - q$$

and

$$\|\lambda - \mu\|_{L^\infty} \le C_0|p - q|.$$

Since functions $(z \mapsto \mu(rz))$ are in $\mathcal{O}_q(\bar{\mathbb{D}}, \Omega_2)$ for $1 > r > 0$, we can estimate:

$$P_{\Omega_1} f(q) = P_{\Omega_2} f(q) \le \frac{1}{2\pi}\int_0^{2\pi} f \circ \mu\big(e^{it}\big)dt \le \frac{1}{2\pi}\int_0^{2\pi}\big(f \circ \lambda\big(e^{it}\big) + \omega(C_0\delta)\big)dt \le P_{\Omega_1} f(p) + \omega(C_0\delta) + \varepsilon.$$

Letting ε to 0, we can conclude that $P_{\Omega_1} f$ is uniformly continuous, with a modulus of continuity $\tilde{\omega}(x) = \omega(C_0 x)$.

2.3. Kuzman–Poletsky theorem

For a domain $\Omega \subset M = \mathbb{C}^n$ and an upper semicontinuous function f, Poletsky (see [9]) proved that $P_\Omega f$ is a plurisubharmonic function (moreover it is the largest plurisubharmonic minorant of f). The key tool in the proof of Theorem 1 is a result of Kuzman, who showed the same for any 2-dimensional almost complex manifold (see Theorem 1 in [7]). The only reason for the assumption about a dimension in our theorem is just this assumption in Kuzman's theorem.

Proof of Theorem 1. The theorem is local, hence we can assume that $P \in \mathbb{C}^2$ and J is $\mathcal{C}^1$ close to J_{st}. We can choose a neighborhood D of P such that there exists a positive continuous strictly J-plurisubharmonic[3] exhaustion function ρ on D.[4] Set $u \in \mathcal{PSH}(D)$. Let us take a decreasing sequence of continuous functions ϕ_k tending to u. We can modify ρ such that $\lim_{z\to\partial D}(\rho - \phi_1) = +\infty$ and put $\tilde{\phi}_k = \max\{\phi_k, \rho - k\}$. There are domains $D_k \Subset D$ such that $\tilde{\phi}_k = \rho - k$ on some neighborhood U_k of $D \setminus D_k$. By Kuzman's result $\hat{\phi}_k = P_D\tilde{\phi}_k$, $P_{D_k}\tilde{\phi}_k$ are J-plurisubharmonic. Note that $\hat{\phi}_k = \rho - k$ on U_k and $P_{D_k}\tilde{\phi}_k = \rho - k$ on $D_k \cap U_k$, hence by Lemma 4 $\hat{\phi}_k \in \mathcal{C}(D)$. Thus we get a decreasing sequence of continuous J-plurisubharmonic functions $\hat{\phi}_k$ tending to u.

By the Richberg theorem (see Theorem 3.1 in [8]), there are functions $\psi_k \in \mathcal{C}^\infty \cap \mathcal{PSH}(D)$ such that

$$\hat{\phi}_k + 2^{-k-1}\rho \le \psi_k \le \hat{\phi}_k + 2^{-k}\rho$$

and we can see that a sequence ψ_k decreases to u.

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[3] "A function u is strictly plurisubharmonic on D" means as usually that for any $\varphi \in \mathcal{C}_0^2$ there is $\varepsilon > 0$ such that $u + \varepsilon\varphi$ is plurisubharmonic. We write here J-plurisubharmonic instead of plurisubharmonic to stress that a function is plurisubharmonic with respect to the almost complex structure J (note that on D we have also the almost complex structure J_{st}).

[4] Such domain D is called a Stain domain, see [2]. Here we can take $D = \{|z - P| < \varepsilon\}$ and $\rho = -\log(\varepsilon - |z - P|^2)$ for $\varepsilon > 0$ small enough.