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Dynamical systems

On the resurgent approach to Écalle–Voronin's invariants

Approche résurgente des invariants d'Écalle–Voronin

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ABSTRACT

Given a holomorphic germ at the origin of $\mathbb{C}$ with a simple parabolic fixed point, the Fatou coordinates have a common asymptotic expansion whose formal Borel transform is resurgent. We show how to use Écalle's alien operators to study the singularities in the Borel plane and relate them to the horn maps, providing each of Écalle–Voronin's invariants in the form of a convergent numerical series. The proofs are original and self-contained, with ordinary Borel summability as the only prerequisite.

RÉSUMÉ

Un germe parabolique simple admet une paire de coordonnées de Fatou qui ont la même série asymptotique résurgente. Nous montrons comment utiliser les opérateurs étrangers d'Écalle pour étudier les singularités dans le plan de Borel et les relier aux applications de corne, de façon à obtenir chaque invariant d'Écalle–Voronin comme une série numérique géométriquement convergente.

Version française abrégée

Pour $\beta \in \mathbb{C}$, de partie réelle > -1, on note $\mathcal{B}: \tilde{\phi} \in z^{-\beta-1}\mathbb{C}[\![z^{-1}]\!] \mapsto \hat{\phi} \in \zeta^{\beta}\mathbb{C}[\![\zeta]\!]$ la transformation de Borel formelle. On dit que la série $\tilde{\phi}$ est $2\pi\mathrm{i}\mathbb{Z}$-résurgente ou que la série $\hat{\phi}$ est $2\pi\mathrm{i}\mathbb{Z}$-résurgente si $\hat{\phi}$ converge pour $|\zeta| < 2\pi$ et se prolonge analytiquement le long de tout chemin issu de 1 contenu dans $\mathbb{C} \setminus 2\pi\mathrm{i}\mathbb{Z}$. Si un tel chemin Γ aboutit en $\omega + 1$, où $\omega \in 2\pi\mathrm{i}\mathbb{Z}$, et que le germe holomorphe en $\omega + 1$ obtenu par prolongement analytique le long de Γ vérifie $\operatorname{cont}_{\Gamma}\hat{\phi}(\omega+\zeta) = \frac{c}{2\pi\mathrm{i}\zeta} + \hat{\chi}(\zeta)\frac{\log\zeta}{2\pi\mathrm{i}} + \hat{R}(\zeta)$ pour $\zeta > 0$, avec $c \in \mathbb{C}$ et $\hat{\chi}, \hat{R} \in \mathbb{C}\{\zeta\}$, on dit que $\tilde{\phi}$ ou $\hat{\phi}$ est (ω, Γ)-simple et on pose $\hat{\mathscr{A}}^{\Gamma}_{\omega}\hat{\phi} := \hat{\chi}$ (monodromie autour de 0 de $\zeta \mapsto \operatorname{cont}_{\Gamma}\hat{\phi}(\omega+\zeta)$) et $\mathscr{A}^{\Gamma}_{\omega}\tilde{\phi} := c + \mathcal{B}^{-1}\hat{\chi} \in \mathbb{C}[\![z^{-1}]\!]$.

Considérons un germe parabolique simple à l'infini avec les notations de [1] : $f(z) = z + 1 + b(z+1) = z + 1 - \rho z^{-1} + O(z^{-2})$, avec $b \in z^{-1}\mathbb{C}\{z^{-1}\}$. Il a une paire de *coordonnées de Fatou* $v^{\pm}_{*}(z) = z + \rho\log z + \mathcal{L}^{\pm}\hat{\varphi}(z)$, obtenues, l'une pour $\arg z \in (-\pi, \pi)$, l'autre pour $\arg z \in (0, 2\pi)$, par ressommation de l'unique solution formelle $\tilde{\varphi} \in z^{-1}\mathbb{C}[\![z^{-1}]\!]$ de l'équation

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$C_{\mathrm{id}-1}\tilde{\varphi} = C_{\mathrm{id}+b}\tilde{\varphi} + b_*$, avec $b_* := b + \rho \log \frac{1+z^{-1}b(z)}{1-z^{-1}} \in z^{-2}\mathbb{C}\{z^{-1}\}$ et $C_g :=$ opérateur de composition avec g. Les nombres A_m, $m \in \mathbb{Z}^*$ définis par

$$v_*^+ \circ \left(v_*^-\right)^{-1}(Z) = Z + \sum_{m\geq 1} A_m \mathrm{e}^{2\pi \mathrm{i} m Z} \quad \text{pour } \operatorname{Im} Z \gg 0,$$

$$v_*^+ \circ \left(v_*^-\right)^{-1}(Z) = Z - 2\pi \mathrm{i}\rho + \sum_{m\geq 1} A_{-m} \mathrm{e}^{-2\pi \mathrm{i} m Z} \quad \text{pour } \operatorname{Im} Z \ll 0,$$

sont les *invariants d'Écalle–Voronin* de f [2,8,1].

Notation. Pour $\operatorname{Re}\beta > -1$, on note E_β l'opérateur réciproque de $C_{\mathrm{id}-1} - \mathrm{Id}\colon z^{-\beta-1}\mathbb{C}[\![z^{-1}]\!] \to z^{-\beta-2}\mathbb{C}[\![z^{-1}]\!]$. Pour $\alpha, \omega \in \mathbb{C}$, on définit $B^\omega\colon \tilde{\psi} \mapsto \mathrm{e}^{-\omega b_*} C_{\mathrm{id}+b}\tilde{\psi} - \tilde{\psi}$ et $B_\alpha\colon \tilde{\phi} \in z^{-\beta-1}\mathbb{C}[\![z^{-1}]\!] \mapsto c_\alpha C_{\mathrm{id}+b}\tilde{\phi} - \tilde{\phi} \in z^{-\beta-2}\mathbb{C}[\![z^{-1}]\!]$, où $c_\alpha := (\frac{1+z^{-1}b}{1-z^{-1}})^\alpha$. Pour $N \in \mathbb{N}$ on définit $\{\tilde{\varphi}\}_N \in z^{-N-1}\mathbb{C}[\![z^{-1}]\!]$ par $\tilde{\varphi} = [\tilde{\varphi}]_N + \{\tilde{\varphi}\}_N$ avec $[\tilde{\varphi}]_N \in \operatorname{Span}(z^{-1}, z^{-2}, \ldots, z^{-N})$.

Théorème 1. *Soient $\omega \in 2\pi \mathrm{i}\mathbb{Z}$ et Γ comme ci-dessus. Soient $\alpha := -\rho\omega$, $\beta := \alpha + N$, N entier $\geq \max\{0, -2\operatorname{Re}\alpha\}$. Pour tout $k \in \mathbb{N}$, $\tilde{\Phi}_k(z) := (E_\beta B_\alpha)^k E_\beta b_\alpha \in z^{-\beta-1}\mathbb{C}[\![z^{-1}]\!]$ est (ω, Γ)-simple ; pour tout chemin γ de $\mathbb{C} \setminus 2\pi \mathrm{i}\mathbb{Z}$ issu de 1, la série $\sum_{k\in\mathbb{N}} \operatorname{cont}_\gamma \mathscr{B}\tilde{\Phi}_k$ converge géométriquement et a pour somme $\operatorname{cont}_\gamma \mathscr{B}(z^{-\alpha}\{\tilde{\varphi}\}_N)$; il existe une série numérique géométriquement convergente $S_\omega^\Gamma = \sum S_{\omega,k}^\Gamma$ telle que, pour tout $k \in \mathbb{N}$, $\mathscr{A}_\omega^\Gamma \tilde{\Phi}_k = \sum_{k_1+k_2=k} S_{\omega,k_1}^\Gamma (E_0 B^\omega)^{k_2} 1$ et $\mathscr{A}_\omega^\Gamma(z^{-\alpha}\{\tilde{\varphi}\}_N) = S_\omega^\Gamma \mathrm{e}^{-\omega\tilde{\varphi}}$.*

Addendum. *La fonction $K_\alpha(\xi,\zeta) := \hat{c}_*(\zeta - \xi) + \sum_{k\geq 1} \frac{(-\xi)^k}{k!} \hat{b}^{*k}(\zeta - \xi) + \sum_{k\geq 1} \frac{(-\xi)^k}{k!} (\hat{c}_* * \hat{b}^{*k})(\zeta - \xi)$, où $\hat{c}_* := \mathscr{B}(c_\alpha - 1)$, est holomorphe sur $\mathbb{C} \times \mathbb{C}$ et*

$$S_{\omega,0}^\Gamma = 2\pi \mathrm{i} \operatorname{cont}_{\tilde{\Gamma}} \hat{b}_\alpha(\omega),$$

$$S_{\omega,k}^\Gamma = 2\pi \mathrm{i} \int_{\Delta_{\tilde{\Gamma},k}} \operatorname{cont}_{\tilde{\Gamma}} \hat{b}_\alpha(\xi_1) \frac{K_\alpha(\xi_1,\xi_2)\cdots K_\alpha(\xi_{k-1},\xi_k) K_\alpha(\xi_k,\omega)}{(\mathrm{e}^{\xi_1}-1)\cdots(\mathrm{e}^{\xi_{k-1}}-1)(\mathrm{e}^{\xi_k}-1)} \mathrm{d}\xi_1 \wedge \cdots \wedge \mathrm{d}\xi_k, \quad k \geqslant 1,$$

avec la notation $\Delta_{\tilde{\Gamma},k} := \{(\tilde{\Gamma}(s_1), \ldots, \tilde{\Gamma}(s_k)) \mid s_1 \leqslant \cdots \leqslant s_k\}$ pour chaque $k \geqslant 1$, $\tilde{\Gamma}$ désignant une paramétrisation du chemin obtenu par concaténation du segment $(0,1]$, du chemin Γ et du segment $[\omega+1, \omega]$.

Théorème 2. *On a $A_{-m} = S_{2\pi \mathrm{i} m}^{\Gamma_m} \mathrm{e}^{-4\pi^2 m\rho}$ pour $m > 0$ et $A_{-m} = -S_{2\pi \mathrm{i} m}^{\Gamma_m}$ pour $m < 0$, où $\Gamma_m := [1, 2\pi \mathrm{i} m + 1]$.*

1. Alien operators for simple $2\pi \mathrm{i}\mathbb{Z}$-resurgent series

We first present an extension of the classical Borel–Laplace summation theory [4,3,7], to be applied to the formal Fatou coordinate of a simple parabolic germ in next section.

We denote by $\mathbb{C}_{\log}$ the Riemann surface of the logarithm viewed as the universal cover of $\mathbb{C}^*$ with base-point at 1. For $\beta_1, \beta_2 \in \mathbb{C}$ with $\operatorname{Re}\beta_1, \operatorname{Re}\beta_2 > -1$ and $\hat{\phi}_1 \in \zeta^{\beta_1}\mathbb{C}\{\zeta\}$, $\hat{\phi}_2 \in \zeta^{\beta_2}\mathbb{C}\{\zeta\}$, we extend the usual convolution and define

$$\hat{\phi}_1 * \hat{\phi}_2 \in \zeta^{\beta_1+\beta_2+1}\mathbb{C}\{\zeta\}, \qquad \hat{\phi}_1 * \hat{\phi}_2(\zeta) := \int_0^1 \hat{\phi}_1\big((1-t)\zeta\big)\hat{\phi}_2(t\zeta)\zeta \,\mathrm{d}t \quad \text{for } \zeta \in \mathbb{C}_{\log} \text{ with } |\zeta| \text{ small enough.}$$

For $\beta \in \mathbb{C}$ with $\operatorname{Re}\beta > -1$, we extend the classical formal Borel transform by defining

$$\mathscr{B}\colon \tilde{\phi} = \sum_{n\geq 0} c_n z^{-n-\beta-1} \in z^{-\beta-1}\mathbb{C}\left[\!\left[z^{-1}\right]\!\right] \mapsto \hat{\phi} := \sum_{n\geq 0} c_n \frac{\zeta^{n+\beta}}{\Gamma(n+\beta+1)} \in \zeta^\beta\mathbb{C}[\![\zeta]\!].$$

Observe that if $\mathscr{B}\tilde{\phi}_1 = \hat{\phi}_1 \in \zeta^{\beta_1}\mathbb{C}\{\zeta\}$ and $\mathscr{B}\tilde{\phi}_2 = \hat{\phi}_2 \in \zeta^{\beta_2}\mathbb{C}\{\zeta\}$ then $\mathscr{B}(\tilde{\phi}_1\tilde{\phi}_2) = \hat{\phi}_1 * \hat{\phi}_2$.

Definition 1.

- Given $\beta \in \mathbb{C}$ with $\operatorname{Re}\beta > -1$, if $\hat{\phi} \in \zeta^\beta\mathbb{C}\{\zeta\}$ converges for any ζ on $\mathbb{C}_{\log}$ with $|\zeta| < 2\pi$ and extends analytically along any path of $\mathbb{C} \setminus 2\pi \mathrm{i}\mathbb{Z}$ issuing from 1, then $\hat{\phi}$ and $\tilde{\phi} = \mathscr{B}^{-1}\hat{\phi}$ are said to be $2\pi \mathrm{i}\mathbb{Z}$-*resurgent*.
- Given $\omega \in 2\pi \mathrm{i}\mathbb{Z}$ and a path Γ in $\mathbb{C} \setminus 2\pi \mathrm{i}\mathbb{Z}$ going from 1 to $\omega + 1$, if $\hat{\phi}$ is $2\pi \mathrm{i}\mathbb{Z}$-resurgent and if its analytic continuation along Γ, which is a holomorphic germ $\operatorname{cont}_\Gamma \hat{\phi}$ at $\omega + 1$, takes the form

$$\operatorname{cont}_\Gamma \hat{\phi}(\omega + \zeta) = \frac{c}{2\pi \mathrm{i}\zeta} + \hat{\chi}(\zeta)\frac{\log\zeta}{2\pi \mathrm{i}} + \hat{R}(\zeta) \quad \text{for } \zeta > 0, \tag{1}$$

where $c \in \mathbb{C}$ and $\hat{\chi}, \hat{R} \in \mathbb{C}\{\zeta\}$, then $\hat{\phi}$ and $\tilde{\phi} = \mathscr{B}^{-1}\hat{\phi}$ are said to be (ω, Γ)-*simple*.

– In the above situation, we set $\hat{\mathscr{A}}^{\Gamma}_{\omega}\tilde{\phi} := \hat{\chi}$, which is the monodromy of $\zeta \mapsto \mathrm{cont}_{\Gamma}\,\hat{\phi}(\omega+\zeta)$ around 0, and $\mathscr{A}^{\Gamma}_{\omega}\tilde{\phi} := c + \mathscr{B}^{-1}\hat{\chi} \in \mathbb{C}[\![z^{-1}]\!]$. The operator $\mathscr{A}^{\Gamma}_{\omega}$ thus defined on the space of all (ω,Γ)-simple formal series is called the *alien operator associated with* (ω,Γ).

(Observe that $\hat{\mathscr{A}}^{\Gamma}_{\omega}\tilde{\phi}$ is itself $2\pi\mathrm{i}\mathbb{Z}$-resurgent because $\mathrm{cont}_{\Gamma}\,\hat{\phi}(\omega+\zeta)$ extends analytically along any path of $\mathbb{C}\setminus 2\pi\mathrm{i}\mathbb{Z}$.)

Definition 2. Let $\hat{\phi}$ be $2\pi\mathrm{i}\mathbb{Z}$-resurgent. We say that $\hat{\phi}$ is of *finite exponential type in non-vertical directions* if, for any path γ of $\mathbb{C}\setminus 2\pi\mathrm{i}\mathbb{Z}$ going from 1 to $\zeta_* \in \mathrm{i}\mathbb{R}$ and any $\delta_0 \in (0,\pi/2)$, there exist $C_0, R_0 > 0$ such that

$$\left|\mathrm{cont}_{\gamma}\,\hat{\phi}\big(\zeta_* + t\,\mathrm{e}^{\mathrm{i}\theta}\big)\right| \le C_0\,\mathrm{e}^{R_0 t} \quad \text{for all } t \ge 0 \text{ and } \theta \in [-\delta_0,\delta_0]\cup[\pi-\delta_0,\pi+\delta_0]. \tag{2}$$

We omit the proof of the following technical statement.

Lemma 1. *Let $\tilde{\phi}_0 \in z^{-1}\mathbb{C}[\![z^{-1}]\!]$ be $2\pi\mathrm{i}\mathbb{Z}$-resurgent with $\hat{\phi}_0 := \mathscr{B}\tilde{\phi}_0$ of finite exponential type in non-vertical directions. Let $\tilde{\phi} := z^{-\alpha}\tilde{\phi}_0 \in z^{-\beta-1}\mathbb{C}[\![z^{-1}]\!]$, where $\alpha,\beta \in \mathbb{C}$ and $\mathrm{Re}\,\beta > -1$. Then $\hat{\phi} := \mathscr{B}\tilde{\phi} \in \zeta^{\beta}\mathbb{C}\{\zeta\}$. Moreover, for any simple curve $\gamma\colon(0,+\infty)\to\mathbb{C}\setminus 2\pi\mathrm{i}\mathbb{Z}$ of the form $\gamma(t) = tu_0$ for $t < t_0$ and $\gamma(t) = c + tu_1$ for $t > t_1$, with $|u_0| = |u_1| = 1$, $u_1 \ne \pm\mathrm{i}$, $c \in \mathbb{C}$ and $0 < t_0 < t_1$, and for any lift $\tilde{\gamma}$ of γ in $\mathbb{C}_{\log}$, the germ $\hat{\phi}$ admits analytic continuation along $\tilde{\gamma}$ and*

$$\int_{\tilde{\gamma}} \mathrm{e}^{-z\zeta}\,\hat{\phi}(\zeta)\,\mathrm{d}\zeta = z^{-\alpha}\int_{\gamma} \mathrm{e}^{-z\zeta}\,\hat{\phi}_0(\zeta)\,\mathrm{d}\zeta \quad \text{for all } z \in \mathbb{C}_{\log}$$
$$\text{with } \arg z \in \left(-\theta-\frac{\pi}{2}, -\theta+\frac{\pi}{2}\right) \text{ and } \mathrm{Re}\big(z\mathrm{e}^{\mathrm{i}\theta}\big) \text{ large enough}, \tag{3}$$

where $\theta \in \mathbb{R}$ is the argument of u_1 which determines the asymptotic direction of $\tilde{\gamma}$.

In particular, $\mathcal{L}^{\theta}\hat{\phi}(z) := \int_0^{\mathrm{e}^{\mathrm{i}\theta}\infty} \mathrm{e}^{-z\zeta}\,\hat{\phi}(\zeta)\,\mathrm{d}\zeta$ is well defined and coincides with $z^{-\alpha}\mathcal{L}^{\theta}\hat{\phi}_0(z)$ for $z \in \mathbb{C}_{\log}$ as above.

Lemma 2. *Let $\theta \in (-\frac{\pi}{2},\frac{\pi}{2})\cup(\frac{\pi}{2},\frac{3\pi}{2})$ and denote by $\mathcal{H}_\theta$ a θ-rotated Hankel contour, i.e. a contour in $\mathbb{C}_{\log}$ which goes along a ray from $\mathrm{e}^{\mathrm{i}(\theta-2\pi)}\infty$ to $\mathrm{e}^{\mathrm{i}(\theta-2\pi)}\varepsilon$ (with $0<\varepsilon<\pi$), circles counterclockwise 0 and then follows the ray from $\mathrm{e}^{\mathrm{i}\theta}\varepsilon$ to $\mathrm{e}^{\mathrm{i}\theta}\infty$. Suppose that $\hat{\phi}$ is (ω,Γ)-simple and $\zeta \mapsto \mathrm{cont}_{\Gamma}\,\hat{\phi}(\omega+\zeta)$ has at most exponential growth along $\mathcal{H}_\theta$, then $\int_{\mathcal{H}_\theta} \mathrm{e}^{-z\zeta}\,\mathrm{cont}_{\Gamma}\,\hat{\phi}(\omega+\zeta)\,\mathrm{d}\zeta = c + \mathcal{L}^{\theta}\hat{\chi}(z)$, where $\mathscr{A}^{\Gamma}_{\omega}\mathscr{B}^{-1}\hat{\phi} = c + \mathscr{B}^{-1}\hat{\chi}$, $\arg z \in (-\theta-\frac{\pi}{2}, -\theta+\frac{\pi}{2})$ and $\mathrm{Re}(z\mathrm{e}^{\mathrm{i}\theta}) \gg 0$.*

Proof. The term $\frac{c}{2\pi\mathrm{i}\zeta}$ contributes c, the remainder contributes $\int_{\mathrm{e}^{\mathrm{i}\theta}\varepsilon}^{\mathrm{e}^{\mathrm{i}\theta}\infty} \mathrm{e}^{-z\zeta}\,\hat{\chi}(\zeta)\,\mathrm{d}\zeta + O(\varepsilon|\ln\varepsilon|)$. $\square$

Lemma 3. *If $b_0 \in \mathbb{C}\{z^{-1}\}$ and $\tilde{\phi}$ is $2\pi\mathrm{i}\mathbb{Z}$-resurgent, then the formal series $b_0\tilde{\phi}$ and $C_{\mathrm{id}+b_0}\tilde{\phi} := \tilde{\phi}\circ(\mathrm{id}+b_0)$ are $2\pi\mathrm{i}\mathbb{Z}$-resurgent. If moreover $\tilde{\phi}$ is (ω,Γ)-simple, then they are also (ω,Γ)-simple, with*

$$\mathscr{A}^{\Gamma}_{\omega}(b_0\tilde{\phi}) = b_0\mathscr{A}^{\Gamma}_{\omega}\tilde{\phi}, \qquad \mathscr{A}^{\Gamma}_{\omega}C_{\mathrm{id}+b_0}\tilde{\phi} = \mathrm{e}^{-\omega b_0}C_{\mathrm{id}+b_0}\mathscr{A}^{\Gamma}_{\omega}\tilde{\phi}.$$

Idea of the proof. Start with $b_0 \in z^{-1}\mathbb{C}\{z^{-1}\}$, hence $\hat{b}_0(\zeta)$ is entire; $\tilde{\psi}_1 := b_0\tilde{\phi}$ and $\tilde{\psi}_2 := C_{\mathrm{id}+b_0}\tilde{\phi} - \tilde{\phi}$ have Borel images

$$\hat{\psi}_j(\zeta) = \int_0^{\zeta} K_j(\xi,\zeta)\hat{\phi}(\xi)\,\mathrm{d}\xi \quad \text{for } \zeta \in \mathbb{C}_{\log} \text{ with } |\zeta| < 2\pi,\ j = 1,2, \tag{4}$$

where $K_1(\xi,\zeta) = \hat{b}_0(\zeta-\xi)$ and $K_2(\xi,\zeta) = \sum_{k\ge1}\frac{(-\xi)^k}{k!}\hat{b}_0^{*k}(\zeta-\xi)$ are holomorphic in $\mathbb{C}\times\mathbb{C}$ (cf. the proof of Lemma 2 in [1]). The analytic continuation of $\hat{\psi}_1$ or $\hat{\psi}_2$ along a path $\gamma\colon[0,1]\to\mathbb{C}\setminus 2\pi\mathrm{i}\mathbb{Z}$ is thus given by the same integral as (4), but integrating on the concatenation $[0,\gamma(0)]+\gamma+[\gamma(1),\zeta]$. It is then possible to analyze the singularities of $\hat{\psi}_1$ and $\hat{\psi}_2$ at ω. We omit the details. $\square$

The above is inspired by Écalle's resurgence theory [2]. In fact, $2\pi\mathrm{i}\mathbb{Z}$-resurgent series are stable under multiplication (see [2], [5] or [6]), and so are the (ω,Γ)-simple ones, but we shall require neither these facts in this article, nor the far-reaching extension of the framework, which allows us to define alien operators or alien derivations in much more general situations.

2. The singular structure of the formal Borel transform of the formal Fatou coordinate

We use the same notations as in [1]: $f = (\mathrm{id}+b)\circ(\mathrm{id}+1) = z+1-\rho z^{-1}+O(z^{-2})$ is a simple parabolic germ at ∞; under the change of unknown $v(z) = z+\rho\log z+\varphi(z)$ the equation $v\circ f = v+1$ is transformed into

$$C_{\mathrm{id}-1}\varphi = C_{\mathrm{id}+b}\varphi + b_*, \qquad b_*(z) := b(z)+\rho\log\frac{1+z^{-1}b(z)}{1-z^{-1}} \in z^{-2}\mathbb{C}\{z^{-1}\} \tag{5}$$

which has a unique solution $\tilde\varphi \in z^{-1}\mathbb{C}[\![z^{-1}]\!]$, shown to be $2\pi\mathrm{i}\mathbb{Z}$-resurgent and Borel summable. Setting $\hat\varphi := \mathscr{B}\tilde\varphi$, one gets two normalized Fatou coordinates $v_*^{\pm}(z) = z+\rho\log z+\mathcal{L}^{\pm}\hat\varphi(z)$, where $\mathcal{L}^{+}\hat\varphi(z)$ and $\mathcal{L}^{-}\hat\varphi(z)$ are Laplace transforms along $\mathbb{R}^+$ and $\mathbb{R}^-$, holomorphic for $\arg z\in(-\pi,\pi)$ and $\arg z\in(0,2\pi)$, and a pair of normalized lifted horn maps $h_*^{\mathrm{up/low}} = v_*^+\circ(v_*^-)^{-1}$ (or Écalle–Voronin modulus—see [2], [8], [1]):

$$h_*^{\mathrm{up}}(Z) = Z+\sum_{m\ge1}A_m\mathrm{e}^{2\pi\mathrm{i}mZ} \quad \text{for } \operatorname{Im}Z\gg0, \qquad h_*^{\mathrm{low}}(Z) = Z-2\pi\mathrm{i}\rho+\sum_{m\ge1}A_{-m}\mathrm{e}^{-2\pi\mathrm{i}mZ} \quad \text{for } \operatorname{Im}Z\ll0.$$

We are interested in the Écalle–Voronin invariants A_m, $m\in\mathbb{Z}^*$.

Let $\omega\in2\pi\mathrm{i}\mathbb{Z}^*$ and Γ be a path in $\mathbb{C}\setminus2\pi\mathrm{i}\mathbb{Z}$ going from 1 to $\omega+1$. In general $\tilde\varphi$ is not (ω,Γ)-simple, but we shall directly prove that, up to a linear combination of monomials, $z^{\rho\omega}\tilde\varphi$ is (ω,Γ)-simple. More precisely, we shall prove the (ω,Γ)-simplicity of $\tilde\Phi := z^{\rho\omega}\{\tilde\varphi\}_N$, with the notation

$$\tilde\varphi = [\tilde\varphi]_N+\{\tilde\varphi\}_N, \quad [\tilde\varphi]_N\in\operatorname{Span}\big(z^{-1},z^{-2},\dots,z^{-N}\big),\ \{\tilde\varphi\}_N\in z^{-N-1}\mathbb{C}\big[\!\big[z^{-1}\big]\!\big],$$

where $N\ge\max\{0,-2\operatorname{Re}\alpha\}$ is integer, with $\alpha := -\rho\omega$. Observe that $\tilde\Phi\in z^{-\beta-1}\mathbb{C}[\![z^{-1}]\!]$ with $\beta := \alpha+N$ and $\operatorname{Re}\beta\ge0$. We shall also compute $\mathscr{A}_\omega^\Gamma\tilde\Phi$ and relate it to the lifted horn maps of f.

We first introduce an auxiliary sequence of formal series, using the same operator $E\colon z^{-2}\mathbb{C}[\![z^{-1}]\!]\to z^{-1}\mathbb{C}[\![z^{-1}]\!]$ inverse of $C_{\mathrm{id}-1}-\mathrm{Id}$ as in [1], but replacing B with $B^\omega\colon\tilde\psi\mapsto\mathrm{e}^{-\omega b_*}C_{\mathrm{id}+b}\tilde\psi-\tilde\psi = \mathrm{e}^{-\omega b_*}B\tilde\psi+(\mathrm{e}^{-\omega b_*}-1)\tilde\psi$.

Proposition 1. *The formal series $\tilde\Psi_k = (EB^\omega)^k1\in z^{-k}\mathbb{C}[\![z^{-1}]\!]$ yield a formally convergent series $\sum\tilde\Psi_k$ and $\sum_{k\ge0}\tilde\Psi_k = \mathrm{e}^{-\omega\tilde\varphi}$. For each $k\ge1$, $\tilde\Psi_k$ is $2\pi\mathrm{i}\mathbb{Z}$-resurgent and, for any $\varepsilon, L>0$, there exist $C_0, M>0$ such that*

$$|\operatorname{cont}_\gamma\mathscr{B}\tilde\Psi_k|\le C_0\frac{M^k}{k!}, \quad \textit{for all } k\ge1 \textit{ and } \gamma\in\mathscr{R}_{\varepsilon,L},$$

where $\mathscr{R}_{\varepsilon,L}$ is the set of all paths of length $<L$ issuing from 1 and staying at distance $>\varepsilon$ from $2\pi\mathrm{i}\mathbb{Z}$.

Proof. Let $c^\omega := \mathrm{e}^{-\omega b_*}-1$. The estimates are obtained by adapting the proof of Lemma 2 of [1], with a new kernel function $K^\omega(\xi,\zeta) = \hat c^\omega(\zeta-\xi)+\sum_{k\ge1}\frac{(-\xi)^k}{k!}\hat b^{*k}(\zeta-\xi)+\sum_{k\ge1}\frac{(-\xi)^k}{k!}(\hat c^\omega*\hat b^{*k})(\zeta-\xi)$. The formal series $\tilde\Psi := \sum_{k\ge0}\tilde\Psi_k$ is the unique solution with constant term 1 of $C_{\mathrm{id}-1}\tilde\Psi = \mathrm{e}^{-\omega b_*}C_{\mathrm{id}+b}\tilde\Psi$—so is $\mathrm{e}^{-\omega\tilde\varphi}$, as can be seen by exponentiating (5). □

Since $\operatorname{Re}\beta\ge0$, the operator $C_{\mathrm{id}-1}-\mathrm{Id}$ maps $z^{-\beta-1}\mathbb{C}[\![z^{-1}]\!]$ to $z^{-\beta-2}\mathbb{C}[\![z^{-1}]\!]$ bijectively; we denote by E_β the inverse, whose Borel counterpart is the multiplication operator $\hat E_\beta\colon\hat\phi(\zeta)\in\zeta^{\beta+1}\mathbb{C}[\![\zeta]\!]\mapsto\frac{1}{\mathrm{e}^\zeta-1}\hat\phi(\zeta)\in\zeta^\beta\mathbb{C}[\![\zeta]\!]$. We introduce an operator

$$B_\alpha\colon\tilde\phi\in z^{-\beta-1}\mathbb{C}\big[\!\big[z^{-1}\big]\!\big]\mapsto c_\alpha C_{\mathrm{id}+b}\tilde\phi-\tilde\phi\in z^{-\beta-2}\mathbb{C}\big[\!\big[z^{-1}\big]\!\big], \quad \text{with } c_\alpha := \left(\frac{1+z^{-1}b}{1-z^{-1}}\right)^\alpha\in1+z^{-1}\mathbb{C}\{z^{-1}\}.$$

Theorem 1. *Let $b_\alpha := z^{-\alpha}(1-z^{-1})^{-\alpha}b_N$, with $b_N := b_*-C_{\mathrm{id}-1}[\tilde\varphi]_N+C_{\mathrm{id}+b}[\tilde\varphi]_N$.*

- *The formal series $\tilde\Phi_k(z) := (E_\beta B_\alpha)^kE_\beta b_\alpha\in z^{-\beta-1}\mathbb{C}[\![z^{-1}]\!]$, $k\ge0$, are (ω,Γ)-simple, and so is $z^{-\alpha}\{\tilde\varphi\}_N$.*
- *For any $\varepsilon, L>0$, there exist $C_0, \Lambda>0$ with $\Lambda<1$ such that, for each $\gamma\in\mathscr{R}_{\varepsilon,L}$, $|\operatorname{cont}_\gamma\mathscr{B}\tilde\Phi_k|\le C_0\Lambda^k$. Moreover, $\hat\Phi := \sum_{k\ge0}\mathscr{B}\tilde\Phi_k$ coincides with the formal Borel transform of $z^{-\alpha}\{\tilde\varphi\}_N$.*
- *There exist complex numbers $S^\Gamma_{\omega,k}$, $k\ge0$, which are $O(\Lambda^k)$ for some $\Lambda<1$ and satisfy*

$$\mathscr{A}_\omega^\Gamma\tilde\Phi_k = \sum_{k_1+k_2=k}S^\Gamma_{\omega,k_1}\tilde\Psi_{k_2} \quad \textit{for each } k\ge0, \qquad \mathscr{A}_\omega^\Gamma\big(z^{-\alpha}\{\tilde\varphi\}_N\big) = S^\Gamma_\omega\,\mathrm{e}^{-\omega\tilde\varphi} \quad \textit{where } S^\Gamma_\omega = \sum_{k\ge0}S^\Gamma_{\omega,k}. \tag{6}$$

Theorem 2. *For $m\in\mathbb{Z}^*$, let Γ_m denote the line segment going from 1 to $2\pi\mathrm{i}m+1$. Then the Écalle–Voronin invariants of f are given by $A_{-m} = S^{\Gamma_m}_{2\pi\mathrm{i}m}\mathrm{e}^{-4\pi^2m\rho}$ if $m>0$, $A_{-m} = -S^{\Gamma_m}_{2\pi\mathrm{i}m}$ if $m<0$.*

Remark 1. When $\alpha \neq 0$, the series of formal series $\sum \tilde{\Phi}_k$ is not convergent for the topology of the formal convergence (but it converges for the product topology $z^{-\beta-1}\mathbb{C}[\![z^{-1}]\!] \simeq \mathbb{C}^{\mathbb{N}}$) and, unless $-\alpha \in \mathbb{N}$ or $S^{\Gamma}_{\omega} = 0$, the singularity at ω of $\mathrm{cont}_{\Gamma}\,\mathscr{B}\tilde{\varphi}$ is not "simple": it is rather of the form $\mathrm{cont}_{\Gamma}\,\mathscr{B}\tilde{\varphi}(\omega+\zeta) = \frac{1}{\Gamma(-\alpha)} S^{\Gamma}_{\omega}\zeta^{-\alpha-1}(1+O(|\zeta|)) + O(1)$ if $\alpha \notin \mathbb{Z}$ or $\frac{(-1)^{\alpha}\alpha!}{2\pi\mathrm{i}} S^{\Gamma}_{\omega}\zeta^{-\alpha-1}(1+O(|\zeta|))$ if $\alpha \in \mathbb{N}$.

Remark 2. Theorem 2 can be extracted from [2], which gives detailed proofs only for the case $\rho = 0$, and also the case $\Gamma = \Gamma_m$ of the second part of (6) with the equivalent formulation $\Delta^{+}_{\omega_m}\tilde{\varphi} = A_m z^{-\rho\omega_m}\mathrm{e}^{-\omega_m\tilde{\varphi}}$ ("Bridge equation"), where $\omega_m = 2\pi\mathrm{i}m$. The name "resurgence" evokes the fact that the singular behaviour near ω_m of the analytic continuation of a germ at 0 like $\hat{\Phi}$ can be explicitly expressed in terms of $\hat{\Phi}$ itself.

The representation of $\hat{\Phi} = \mathscr{B}(z^{-\alpha}\{\tilde{\varphi}\}_N)$ as the convergent series $\sum \hat{\Phi}_k = \sum \mathscr{B}((E_\beta B_\alpha)^k E_\beta b_\alpha)$ and the computation of the action of the alien operator $\mathscr{A}^{\Gamma}_{\omega}$ in (6) are new. The point is that $\mathscr{A}^{\Gamma}_{\omega}\tilde{\Phi}_k = S^{\Gamma}_{\omega,k} + O(z^{-1})$, so the first part of (6) says that

$$\mathrm{cont}_{\Gamma}\,\hat{\Phi}_k(\omega+\zeta) = \frac{S^{\Gamma}_{\omega,k}}{2\pi\mathrm{i}\zeta} + \hat{\mathscr{A}}^{\Gamma}_{\omega}\tilde{\Phi}_k(\zeta)\frac{\log\zeta}{2\pi\mathrm{i}} + \text{regular germ}, \qquad \hat{\mathscr{A}}^{\Gamma}_{\omega}\tilde{\Phi}_k = \sum_{k_1+k_2=k,\ k_2\ge 1} S^{\Gamma}_{\omega,k_1}\hat{\Psi}_{k_2} \in \mathbb{C}\{\zeta\}.$$

The numbers $S^{\Gamma}_{\omega,k}$ appear as generalized residua, for which we can give quite explicit formulas:

Addendum to Theorem 1. Let $\hat{c}_* := \mathscr{B}(c_\alpha - 1)$, which is an entire function, and

$$K_\alpha(\xi,\zeta) = \hat{c}_*(\zeta-\xi) + \sum_{k\ge 1}\frac{(-\xi)^k}{k!}\hat{b}^{*k}(\zeta-\xi) + \sum_{k\ge 1}\frac{(-\xi)^k}{k!}\big(\hat{c}_* * \hat{b}^{*k}\big)(\zeta-\xi), \tag{7}$$

which is holomorphic in $\mathbb{C}\times\mathbb{C}$. Let $\tilde{\Gamma}$ denote a parameterization of the path obtained by concatenating the line segment $(0,1]$, the path Γ and the line segment $[\omega+1,\omega]$. Then

$$S^{\Gamma}_{\omega,0} = 2\pi\mathrm{i}\,\mathrm{cont}_{\tilde{\Gamma}}\,\hat{b}_\alpha(\omega),$$
$$S^{\Gamma}_{\omega,k} = 2\pi\mathrm{i}\int_{\Delta_{\tilde{\Gamma},k}} \mathrm{cont}_{\tilde{\Gamma}}\,\hat{b}_\alpha(\xi_1)\frac{K_\alpha(\xi_1,\xi_2)\cdots K_\alpha(\xi_{k-1},\xi_k)K_\alpha(\xi_k,\omega)}{(\mathrm{e}^{\xi_1}-1)\cdots(\mathrm{e}^{\xi_{k-1}}-1)(\mathrm{e}^{\xi_k}-1)}\,\mathrm{d}\xi_1\wedge\cdots\wedge\mathrm{d}\xi_k, \quad k\geqslant 1,$$

with the notation $\Delta_{\tilde{\Gamma},k} := \{(\tilde{\Gamma}(s_1),\dots,\tilde{\Gamma}(s_k)) \mid s_1 \leqslant \cdots \leqslant s_k\}$ for each positive integer k. □

Observe that, according to the second part of (6), each coefficient S^{Γ}_{ω}, and in particular each Écalle–Voronin invariant, can be obtained as the convergent series of these "residua" $S^{\Gamma}_{\omega,k}$.

Proof of Addendum to Theorem 1. The normal convergence of the series (7) follows easily from the estimates available for the convolution of entire functions (see inequality (9) in [1]), and the Borel counterpart of B_α is the integral transform with kernel function K_α, hence

$$\mathrm{cont}_{\gamma}\,\hat{B}_\alpha\hat{\phi}\big(\gamma(s)\big) = \int_0^s K_\alpha\big(\gamma(\sigma),\gamma(s)\big)\,\mathrm{cont}_{\gamma}\,\hat{\phi}\big(\gamma(\sigma)\big)\gamma'(\sigma)\,\mathrm{d}\sigma \tag{8}$$

for any $2\pi\mathrm{i}\mathbb{Z}$-resurgent $\hat{\phi}$ and any path $\gamma\colon(0,\ell]\to\mathbb{C}_{\log}$ (with any $\ell>0$), whose projection onto $\mathbb{C}$ avoids $2\pi\mathrm{i}\mathbb{Z}$ and which starts as $\gamma(s) = su_0$ for $s>0$ small enough (with fixed $u_0\in\mathbb{C}^*$). Writing $\tilde{\Phi}_k = E_\beta(B_\alpha E_\beta)^k b_\alpha$, by repeated use of (8), we obtain an explicit formula for the analytic continuation of its Borel transform: for any ζ close enough to the endpoint of a path γ as above,

$$\mathrm{cont}_{\gamma}\,\hat{\Phi}_0(\zeta) = \frac{1}{\mathrm{e}^{\zeta}-1}\,\mathrm{cont}_{\gamma}\,\hat{b}_\alpha(\zeta),$$
$$\mathrm{cont}_{\gamma}\,\hat{\Phi}_k(\zeta) = \frac{1}{\mathrm{e}^{\zeta}-1}\int_{\Delta_{\gamma,k}}\mathrm{cont}_{\gamma}\,\hat{b}_\alpha(\xi_1)\frac{K_\alpha(\xi_1,\xi_2)\cdots K_\alpha(\xi_{k-1},\xi_k)K_\alpha(\xi_k,\zeta)}{(\mathrm{e}^{\xi_1}-1)\cdots(\mathrm{e}^{\xi_{k-1}}-1)(\mathrm{e}^{\xi_k}-1)}\,\mathrm{d}\xi_1\wedge\cdots\wedge\mathrm{d}\xi_k, \quad k\geqslant 1,$$

with the notation $\Delta_{\gamma,k} := \{(\gamma(s_1),\dots,\gamma(s_k)) \mid s_1\leqslant\cdots\leqslant s_k\}$. The conclusion follows. □

Theorem 1 implies Theorem 2. For all $m \ge 1$ we set $\omega_m := 2\pi\mathrm{i}m$ and $\tilde{\Phi}^{(m)} := z^{-\alpha_m}\{\tilde{\varphi}\}_{N_m}$ with $\alpha_m := -\rho\omega_m$ and $N_m := \max\{0, -2\,\mathrm{Re}\,\alpha_m\}$. By [1] we know that $\hat{\varphi}$ is of finite exponential type in non-vertical directions; we fix $\theta^+ \in (0,\frac{\pi}{2})$, $\theta^- \in$

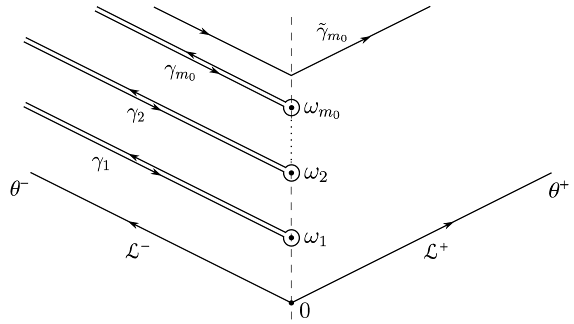

Fig. 1. Deformation of the contour.

$(\frac{\pi}{2},\pi)$ and $Q_R^{\rm low} := \{z \in \mathbb{C}_{\log} \mid \arg z \in (-\theta^+ - \frac{\pi}{2}, -\theta^- + \frac{\pi}{2}), \operatorname{Re}(z\,e^{i\theta^+}) > R, \operatorname{Re}(z\,e^{i\theta^-}) > R\}$, with R large enough. Identifying $z \in Q_R^{\rm low}$ with its projection in $\mathbb{C}$, we have $\operatorname{Im} z < 0$ and the branches of log used in v_*^+ and v_*^- differ by $-2\pi i$; for $m_0 \geq 1$, deforming the contour of integration (see Fig. 1), we get

$$v_*^+(z) - v_*^-(z) = -2\pi i\rho + \big(\mathcal{L}^+ - \mathcal{L}^-\big)\hat\varphi(z) = -2\pi i\rho + \left(\sum_{m=1}^{m_0}\int_{\gamma_m} + \int_{\tilde\gamma_{m_0}}\right) e^{-z\zeta}\hat\varphi(\zeta)\,d\zeta$$

$$= -2\pi i\rho + \sum_{m=1}^{m_0} I_m + O\big(e^{(2\pi m_0+\pi)\operatorname{Im} z}\big),$$

with $I_m = \int_{\gamma_m} e^{-z\zeta}\operatorname{cont}_{\Gamma_m}\hat\varphi(\zeta)\,d\zeta = \int_{\gamma_m} e^{-z\zeta}\operatorname{cont}_{\Gamma_m}\{\hat\varphi(\zeta)\}_{N_m}\,d\zeta$. Since each γ_m can be expressed as the difference between two paths to which Lemma 1 applies, we have $I_m = z^{\alpha_m}\int_{\gamma_m} e^{-z\zeta}\operatorname{cont}_{\Gamma_m}\hat\Phi^{(m)}(\zeta)\,d\zeta$. Denoting by $\mathcal{H}^-$ a θ^--rotated Hankel contour, we get $I_m = z^{\alpha_m}e^{-\omega_m z}\int_{\mathcal{H}^-} e^{-z\zeta}\operatorname{cont}_{\Gamma_m}\hat\Phi^{(m)}(\omega_m+\zeta)\,d\zeta = S_{\omega_m}^{\Gamma_m} e^{\alpha_m \log z - \omega_m z - \omega_m \mathcal{L}^-\hat\varphi}$ by Lemma 2 and the second part of (6) (the Borel–Laplace summation commutes with exponentiation—see e.g. [6]).

Thus

$$\big(h_*^{\rm low} - {\rm id}\big)\circ v_*^- = v_*^+ - v_*^- = -2\pi i\rho + \sum_{m=1}^{m_0} S_{\omega_m}^{\Gamma_m} e^{-\omega_m(-2\pi i\rho + v_*^-)} + O\big(e^{(2\pi m_0+\pi)\operatorname{Im} z}\big) \quad \text{on } Q_R^{\rm low},$$

whence $A_{-m} = S_{\omega_m}^{\Gamma_m} e^{2\pi i\rho\omega_m}$ for $m > 0$. The case $m < 0$ is similar, except that, on the domain $Q_R^{\rm up}$ defined analogously with $-\pi < \theta^- < -\frac{\pi}{2} < \theta^+ < 0$, the two branches of log used in $v_*^{\pm}$ match and the orientation of the paths differs. □

Proof of Theorem 1. First observe that $b_\alpha \in z^{-\beta-2}\mathbb{C}\{z^{-1}\}$ because b_* and $[\tilde\varphi]_N \in \mathbb{C}\{z^{-1}\}$ hence $b_N \in \mathbb{C}\{z^{-1}\}$, and $b_N = C_{{\rm id}-1}\{\tilde\varphi\}_N - C_{{\rm id}+b}\{\tilde\varphi\}_N \in z^{-N-2}\mathbb{C}[[z^{-1}]]$. Thus the formal series $\tilde\Phi_k$ are well defined in $z^{-\beta-1}\mathbb{C}[[z^{-1}]]$; they are $2\pi i\mathbb{Z}$-resurgent because this property is preserved by E_β and B_α (Lemma 3) and we start with $\mathscr{B}\tilde\Phi_0 = \hat E_\beta \hat b_\alpha = \frac{\hat b_\alpha(\zeta)}{e^\zeta - 1}$ meromorphic on $\mathbb{C}_{\log}$. Their (ω,Γ)-simplicity and the existence of constants $S_{\omega,k}^\Gamma$ such that the first part of (6) results by induction on k (using the fact that $B^\omega\tilde\Psi_k \in z^{-2}\mathbb{C}[[z^{-1}]]$ for all k) from Lemma 4.

Lemma 4. *If $\tilde\phi$ is (ω,Γ)-simple, then so is $B_\alpha\tilde\phi$, with $\mathscr{A}_\omega^\Gamma B_\alpha\tilde\phi = B^\omega\mathscr{A}_\omega^\Gamma\tilde\phi$. If moreover $\mathscr{A}_\omega^\Gamma\tilde\phi \in z^{-2}\mathbb{C}[[z^{-1}]]$ (i.e. $c = \hat\chi(0) = 0$ in (1)), then also $E_\beta\tilde\phi$ is (ω,Γ)-simple and $\mathscr{A}_\omega^\Gamma E_\beta\tilde\phi - E\mathscr{A}_\omega^\Gamma\tilde\phi$ is a constant.*

Proof of Lemma 4. The first identity results from Lemma 3 and $e^{-\omega b}c_\alpha = e^{-\omega b_*}$. For the second one, in view of (1), the analytic continuation of $\mathscr{B}E_\beta\tilde\phi$ is $\operatorname{cont}_\Gamma \hat E_\beta\hat\phi(\omega+\zeta) = \frac{\hat\chi(\zeta)}{e^\zeta-1}\frac{\log\zeta}{2\pi i} + \frac{\hat R(\zeta)}{e^\zeta-1}$, hence $\mathscr{A}_\omega^\Gamma E_\beta\tilde\phi = E\mathscr{B}^{-1}\hat\chi + 2\pi i\hat R(0)$. □

We finish the proof of Theorem 1 by estimating $\hat\Phi_k := \mathscr{B}\tilde\Phi_k$ by means of the recursive formula $\hat\Phi_k = \hat E_\beta\hat B_\alpha\hat\Phi_{k-1}$.

Lemma 5. *For $0 < \varepsilon < 1$, $L > 1$, let $\tilde{\mathscr{R}}_{\varepsilon,L}$ denote the set of all naturally parameterised paths $\gamma\colon (0,\ell(\gamma)] \to \mathbb{C}_{\log}$, where $\ell(\gamma) < L$ is the length of γ, for which there exists $\theta \in [-\pi, 3\pi]$ such that $\gamma(s) = s\,e^{i\theta}$ for $s \leq \varepsilon$ and $\operatorname{dist}(\gamma(s), 2\pi i\mathbb{Z}) > \varepsilon$ for $s \geq \varepsilon$. For any integer $\ell \geq 0$, let $V_{\varepsilon,L;\ell}$ denote the space of all $\hat\phi \in \zeta^{\beta+\ell}\mathbb{C}\{\zeta\}$ admitting analytic continuation along the paths of $\tilde{\mathscr{R}}_{\varepsilon,L}$, such that $\|\hat\phi\|_{\varepsilon,L;\ell} := \sup\{s^{-\operatorname{Re}\beta-\ell}|\operatorname{cont}_\gamma\hat\phi(\gamma(s))|$, for $\gamma \in \tilde{\mathscr{R}}_{\varepsilon,L}$ and $s \in (0,\ell(\gamma)]\} < \infty$. Then the operator $\hat E_\beta\hat B_\alpha$ leaves invariant the spaces $V_{\varepsilon,L;\ell}$ and, for each ε and L, there exists $M(\varepsilon,L) > 0$ such that*

$$\|\hat E_\beta\hat B_\alpha\hat\phi\|_{\varepsilon,L;\ell} \leq \frac{M(\varepsilon,L)}{\operatorname{Re}\beta+\ell+1}\|\hat\phi\|_{\varepsilon,L;\ell} \quad \textit{for all } \hat\phi \in V_{\varepsilon,L;\ell} \textit{ and } \ell \geq 0.$$

Proof of Lemma 5. One can find $\kappa = \kappa(\varepsilon, L)$ such that, for each $\gamma \in \tilde{\mathscr{R}}_{\varepsilon,L}$ and $s \in (0, \ell(\gamma)]$, $|\gamma(s)| \geq \kappa s$ (because $\arg \gamma(s)$ is uniformly bounded). Let $M_0 = M_0(\varepsilon, L) := \sup\{|\frac{\zeta}{e^\zeta - 1}|,$ for $|\zeta| < L, \operatorname{dist}(\zeta, 2\pi i\mathbb{Z}^*) > \varepsilon\}$. From (8), we get $\|\hat{B}_\alpha \hat{\phi}\|_{\varepsilon,L;\ell+1} \leq \frac{N_L}{\operatorname{Re}\beta+\ell+1} \|\hat{\phi}\|_{\varepsilon,L;\ell}$ with $N_L := \sup_{|\xi|,|\zeta|<L}\{|K(\xi,\zeta)|\}$, and $\|\hat{E}_\beta \hat{B}_\alpha \hat{\phi}\|_{\varepsilon,L;\ell} \leq \frac{N_L M_0}{\kappa(\operatorname{Re}\beta+\ell+1)} \|\hat{\phi}\|_{\varepsilon,L;\ell}$. □

We have $|\frac{\alpha}{\alpha+N+1}| < 1$ because $\operatorname{Re}\alpha > -\frac{N+1}{2}$. Let us choose an integer $d \geq 0$ so that $\frac{M(\varepsilon,L)}{\operatorname{Re}\beta+d+1} \leq \Lambda := \max\{|\frac{\alpha}{\alpha+N+1}|, \frac{1}{2}\}$. For any $\hat{\phi} \in \zeta^\beta \mathbb{C}\{\zeta\}$, we use the notation:

$$\hat{\phi} = [\hat{\phi}]_d + \{\hat{\phi}\}_d, \quad [\hat{\phi}]_d \in \mathscr{E}_d := \operatorname{Span}\big(\zeta^\beta, \zeta^{\beta+1}, \dots, \zeta^{\beta+d-1}\big), \ \{\hat{\phi}\}_d \in \zeta^{\beta+d}\mathbb{C}\{\zeta\}.$$

Now $\hat{\phi} \in \zeta^{\beta+d}\mathbb{C}\{\zeta\} \Rightarrow \hat{B}_\alpha \hat{\phi} = \alpha 1 * \hat{\phi} + O(\zeta^{\beta+d+2}) \Rightarrow \hat{E}_\beta \hat{B}_\alpha \hat{\phi} = \frac{\alpha}{\zeta} 1 * \hat{\phi} + O(\zeta^{\beta+d+1}) \in \zeta^{\beta+d}\mathbb{C}\{\zeta\}$, thus

$$[\hat{\Phi}_k]_d = A[\hat{\Phi}_{k-1}]_d, \qquad \{\hat{\Phi}_k\}_d = \big\{\hat{E}_\beta \hat{B}_\alpha [\hat{\Phi}_{k-1}]_d\big\}_d + \hat{E}_\beta \hat{B}_\alpha \{\hat{\Phi}_{k-1}\}_d,$$

where $A \in \operatorname{End}\mathscr{E}_d$ is defined by $A\hat{\phi} := [\hat{E}_\beta \hat{B}_\alpha \hat{\phi}]_d$ and has a triangular matrix in the basis $(\zeta^\beta, \zeta^{\beta+1}, \dots, \zeta^{\beta+d-1})$, with $A\zeta^{\beta+\ell} = \lambda_\ell \zeta^{\beta+\ell}(1 + O(\zeta))$, $\lambda_\ell := \frac{\alpha}{\beta+\ell+1}$, $\ell = 0, \dots, d-1$. By the choice of β, the eigenvalues have modulus $|\lambda_\ell| \leq \Lambda$, hence $\|[\hat{\Phi}_k]_d\| = O(\Lambda^k)$ for any norm on $\mathscr{E}_d$. Since $\hat{\phi} \in \mathscr{E}_d \mapsto \{\hat{E}_\beta \hat{B}_\alpha \hat{\phi}\}_d \in V_{\varepsilon,L;d}$ is linear, we can find C_0' such that $\|\{\hat{E}_\beta \hat{B}_\alpha [\hat{\Phi}_{k-1}]_d\}_d\|_{\varepsilon,L;d} \leq C_0' \Lambda^k$ for all $k \geq 1$, hence $\|\{\hat{\Phi}_k\}_d\|_{\varepsilon,L;d} \leq C_0' \Lambda^k + \Lambda \|\{\hat{\Phi}_{k-1}\}_d\|_{\varepsilon,L;d}$. We thus get $\|\{\hat{\Phi}_k\}_d\|_{\varepsilon,L;d} \leq (\|\{\hat{\Phi}_0\}_d\|_{\varepsilon,L;d} + C_0' k)\Lambda^k$ for all $k \geq 0$ and, increasing slightly Λ, the desired bounds for $|\operatorname{cont}_\gamma \hat{\Phi}_k|$.

By uniform convergence, we can now define a function $\hat{\Phi} = \sum_{k\geq 0} \hat{\Phi}_k \in \zeta^\beta \mathbb{C}\{\zeta\}$, whose inverse formal Borel transform $\tilde{\Phi} \in z^{-\beta-1}\mathbb{C}[\![z^{-1}]\!]$ satisfies $\tilde{\Phi} = E_\beta b_\alpha + E_\beta B_\alpha \tilde{\Phi}$, whence

$$C_{\mathrm{id}-1}\tilde{\Phi} = c_\alpha C_{\mathrm{id}+b}\tilde{\Phi} + b_\alpha.$$

Multiplying both sides by $z^\alpha(1 - z^{-1})^\alpha$, we find that $\tilde{\psi} := z^\alpha \tilde{\Phi} \in z^{-N-1}\mathbb{C}[\![z^{-1}]\!]$ satisfies $C_{\mathrm{id}-1}\tilde{\psi} = C_{\mathrm{id}+b}\tilde{\psi} + b_N$, an equation of which $\{\tilde{\varphi}\}_N$ is the unique solution in $z^{-1}\mathbb{C}[\![z^{-1}]\!]$, hence $\tilde{\Phi} = z^{-\alpha}\{\tilde{\varphi}\}_N$.

The estimates for the $\hat{\Phi}_k$'s imply similar estimates for the monodromies $\hat{\chi}_k := \hat{\mathscr{A}}_\omega^\Gamma \tilde{\Phi}_k$, and hence for the residues $S_{\omega,k}^\Gamma$ of the functions $\operatorname{cont}_\gamma \hat{\Phi}_k(\omega + \zeta) - \hat{\chi}(\zeta)\frac{\log \zeta}{2\pi i}$. The second part of (6) follows by uniform convergence. □

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