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# A note on the Bergman Kernel

## *Sur le noyau de Bergman*

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ABSTRACT

It is known that the Bergman kernel associated with $L^k$, where $L$ is positive line bundle over a complex compact manifold, has an asymptotic expansion. We give an elementary proof of the fact that the subprincipal term of this expansion is the scalar curvature.



RÉSUMÉ

Il est connu que le noyau de Bergman associé à $L^k$, où $L$ est un fibré en droite positif sur une variété complexe compacte, admet un développement asymptotique. Nous prouvons de manière élémentaire que le terme sous-principal de ce développement est donné par la courbure scalaire.

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## 1. Introduction

Let $M$ be an $n$-dimensional complex compact manifold. Let $L \to M$ be a Hermitian holomorphic line bundle which is positively curved. Let $g$ be the corresponding Riemannian metric of $M$, so $g(X,Y) = i\Theta(L)(X, JY)$, where $\Theta(L)$ is the Chern curvature of $L$ and $J$ is the complex structure of $M$.

For any $k \in \mathbb{N}$, let $E_k = H^0(M, L^k)$ be the space of holomorphic sections of $L^k$. $M$ being compact, $E_k$ is finite dimensional. It has a natural scalar product given by

$$\langle s,t\rangle = \int_M (s,t)\mu, \quad s,t \in E_k$$

where $(s,t)$ is the pointwise scalar product and $\mu = (i\Theta(L))^n/n!$ is the Liouville measure. Introduce an orthonormal basis $s_{k,i}$, $i = 1,\dots,N_k$ of $E_k$. For any $p,q \in M$, let

$$\Pi_k(p,q) = \sum_{i=1}^{N_k} s_{k,i}(p) \otimes \bar{s}_{k,i}(q) \in L_p^k \otimes \bar{L}_q^k$$

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$\Pi_k$ is a holomorphic section of $L \boxtimes \bar{L}^k \to M \times \overline{M}$. It is the Schwartz kernel of the orthogonal projection of $\mathcal{C}^\infty(M, L^k)$ onto $E_k$. It is called the Bergman kernel of $L^k$. The Hermitian structure induces an isomorphism between $L^k \otimes \bar{L}^k$ and the trivial complex line bundle over $M$ so that $\Pi_k(x,x) \in \mathbb{C}$. Let $n$ be the complex dimension of $M$.

**Theorem 1.1.** *For any $p \in M$, we have*

$$\Pi_k(p,p) = \left(\frac{k}{2\pi}\right)^n \left(1 + k^{-1}\frac{\rho(p)}{2} + \mathrm{o}(k^{-1})\right)$$

*where $\rho \in \mathcal{C}^\infty(M, \mathbb{R})$ is the scalar curvature of $g$.*

Actually a stronger result holds: $\Pi_k(p,p)$ has a full asymptotic expansion in negative power of $k$ whose coefficients are given by universal polynomials in the curvature of $g$ and its successive derivatives. The first results on the asymptotics of $\Pi_k$ are due to Bouche [3] and Tian [19]. The existence of the asymptotic expansion has been obtained independently by Catlin [5] and Zelditch [21], who deduced it from the seminal work of Boutet de Monvel and Sjöstrand [4]. An algorithm to compute the coefficients has been given by Lu in [15]. Later, other algorithms have been proposed by the author [6], Dai, Liu and Ma [9], Berman, Berndtsson and Sjöstrand [1] and Ma and Marinescu [17,18]. More recently, a closed formula based on Feynman diagram has been given by Xu [20].

All these works are rather technical and reserved to the specialists. The goal of this note is to provide an elementary proof of Theorem 1.1. This proof is inspired by the survey article of Berndtsson [2], where a simple proof for the leading order term $\Pi_k(p,p) \sim (\frac{k}{2\pi})^n$ was obtained. So the new argument is the elementary computation of the second term. It is not clear whether we can compute the subsequent terms of the asymptotic expansion with the method presented here.

The fact that the subprincipal term is given by the scalar curvature was important in the work of Donaldson on balanced metrics [11] and in the analysis of Berezin–Toeplitz operators [6–8]. The next terms have applications to the extension of Donaldson's work by Fine regarding the quantization of the Mabuchi energy [13]. More precisely, this latter paper is based on the asymptotic expansion of the Toeplitz kernels established by Ma and Marinescu in [17,18] (cf. also [16]).

As the referee pointed out, another short proof of Theorem 1.1 appears in the lecture notes of Donaldson [12].

## 2. Some general estimates

For any section $s$ of a Hermitian bundle over $M$, we denote by $|s| \in \mathcal{C}^\infty(M, \mathbb{R})$ the pointwise norm of $s$ and by $\|s\|$ the square root of the integral of $|s|^2\mu$ over $M$. We introduce a metric on $(T^{0,1}M)^*$ and define, for any $k$, the metrics of $L^k$ and $L^k \otimes (T^{0,1}M)^*$ by tensoring the ones of $L$ and $(T^{0,1}M)^*$. The following theorems are well known. The first one is proved by applying Cauchy formula on a ball of radius $k^{-1}$. The second one is a version of the Kodaira–Hörmander estimates.

**Theorem 2.1.** *For any Hermitian holomorphic line bundle $M \to L$ with $M$ compact, there exists $C > 0$ such that for any $k > 0$, for any $s \in \mathcal{C}^\infty(M, L^k)$ and any $p \in M$, we have:*

$$\big|s(p)\big| \leqslant C\Big(k^{-n}\|s\| + k^{-1}\sup_M |\bar\partial s|\Big).$$

**Theorem 2.2.** *For any Hermitian holomorphic line bundle $M \to L$ that is positively curved, with $M$ compact, there exist $k_0$ and $C > 0$ such that, for any $k \geqslant k_0$, for any $s \in \mathcal{C}^\infty(M, L^k)$ which is orthogonal to $E_k = H^0(M, L^k)$, we have:*

$$\|s\|^2 \leqslant Ck^{-1}\|\bar\partial s\|^2.$$

**Proof.** Let us sketch the proof. Let $\nabla = \partial + \bar\partial$ be the Chern connection of $L^k$. Introduce the Laplacian $\Delta_{k,\ell} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial : \Omega^{0,\ell}(M, L^k) \to \Omega^{0,\ell}(M, L^k)$. As a consequence of the Bochner–Kodaira–Nakano identity, Chapter 4 in [10], there exist $k_0$ and $C > 0$ such that, for any $k \geqslant k_0$ and any $t \in \Omega^{0,1}(M, L^k)$,

$$\langle \Delta_{k,1}t, t\rangle \geqslant k\|t\|^2/C. \tag{1}$$

By Hodge Theorem, we have an orthogonal decomposition

$$\mathcal{C}^\infty\big(M, L^k\big) = E_k \oplus \Delta_{k,0}\big(\mathcal{C}^\infty\big(M, L^k\big)\big).$$

So for any $s \in \mathcal{C}^\infty(M, L^k)$ orthogonal to $E_k$, there exists $t \in \Omega^{0,1}(M, L^k)$ such that $s = \bar\partial^* t$ and $\bar\partial t = 0$. Consequently $\bar\partial s = \Delta_{k,1}t$. By Eq. (1) and the Cauchy–Schwarz inequality, we have $k\|t\|^2 \leqslant C\|\Delta_{k,1}t\|\|t\|$, which implies that $k\|t\| \leqslant C\|\Delta_{k,1}t\|$. Consequently

$$C\|\bar\partial s\|^2 = C\|\Delta_{k,1}t\|^2 \geqslant k\|t\|\|\Delta_{k,1}t\| \geqslant k\langle t, \Delta_{k,1}t\rangle = k\|s\|^2$$

where we have used again the Cauchy–Schwarz inequality. □

## 3. The proof

We have the following characterization of the Bergman kernel on the diagonal.

**Lemma 3.1.** *For any $k \in \mathbb{N}$ and $p \in M$, $\Pi_k(p,p) = \sup_{s \in E_k \setminus \{0\}} \frac{|s(p)|^2}{\|s\|^2}$.*

**Proof.** Since any unitary $s \in E_k$ is contained in an orthonormal basis, $|s(p)|^2 \leqslant \Pi_k(p,p)$.

Conversely, let $(s_{k,i})$ be an orthonormal basis of $E_k$. Let $\xi$ be a unitary vector of $L^k_p$. Define $s = \sum \bar{\lambda}_i s_{k,i} \in E_k$ with $\lambda_i = s_{k,i}(p)/\xi$. Then $|s(p)|^2/\|s\|^2 = \Pi_k(p,p)$. □

Let $p \in M$. Introduce normal coordinates $(z_i)$ centered at $p$. Then $L$ has a local holomorphic frame $\sigma$ defined on an open neighborhood $U$ of $p$ such that the function $\varphi = -2\ln|\sigma|$ satisfies $\varphi = \varphi_4 + \mathcal{O}(|z|^5)$ with

$$\varphi_4 = \sum_{\ell=1}^{n} |z_\ell|^2 + \sum_{|\alpha|,|\beta|=2} \frac{G_{\alpha,\beta}}{\alpha!\beta!} z^\alpha \bar{z}^\beta.$$

Here the $G_{\alpha,\beta}$ are complex numbers. Since $\Theta(L) = \partial\bar{\partial}\varphi$, the scalar curvature satisfies

$$\rho(p) = -\sum_{\ell,q=1}^{n} G_{\ell q,\ell q}. \tag{2}$$

Restricting $U$ if necessary, we have that $\varphi(q) > 0$ for any $q \in U \setminus \{p\}$. Introduce $\psi \in \mathcal{C}^\infty(M)$ such that $\psi = 1$ on a neighborhood of $p$ and the support of $\psi$ is contained in $U$. Extend $\sigma$ to $M$ by setting $\sigma(q) = 0$ for all $q \notin U$. Then for any integer $k$, define $s_k$ as the harmonic part of $\psi\sigma^k$, so that $s_k$ is holomorphic and $s_k - \psi\sigma^k$ is smooth and orthogonal to $E_k$.

We easily check that $|\bar{\partial}(\psi\sigma^k)| \leqslant C_1 e^{-k/C_1}$ uniformly on $M$ for some $C_1 > 0$. By Theorem 2.1 and Theorem 2.2, we obtain that

$$\left| s_k - \psi\sigma^k \right| \leqslant C_2 e^{-k/C_2}$$

uniformly on $M$ for some $C_2 > 0$.

**Lemma 3.2.** *We have that*

$$\left| s_k(p) \right| = 1 + \mathcal{O}\left(e^{-k/C_2}\right) \quad \textit{and} \quad \|s_k\|^2 = \left(\frac{2\pi}{k}\right)^n \left(1 - k^{-1}\frac{\rho(p)}{2} + \mathcal{O}\left(k^{-2}\right)\right).$$

**Proof.** Since $\Theta(L) = \partial\bar{\partial}\varphi$, we have that $\mu = \det(\partial^2\varphi/\partial z_\ell \partial z_m)\mu_{\mathrm{Leb}}$, where

$$\mu_{\mathrm{Leb}} = i^{n^2} \mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_n \wedge \mathrm{d}\bar{z}_1 \wedge \ldots \wedge \mathrm{d}\bar{z}_n.$$

Furthermore $\det(\partial^2\varphi/\partial z_\ell \partial z_m) = d_2 + \mathcal{O}(|z|^3)$ with

$$d_2 = 1 + \sum_{\ell,q,r=1}^{n} G_{\ell q,\ell r} z_q \bar{z}_r.$$

So by the Laplace method, Theorem of 7.7.5 of [14], with $\Delta f = \sum_\ell \partial^2 f/\partial z_\ell \partial \bar{z}_\ell$,

$$\begin{aligned}\int_U |\psi|^2 e^{-k\varphi}\mu &= \left(\frac{2\pi}{k}\right)^n \left(1 + k^{-1}\Delta\left[\sum_{\ell,q,r=1}^{n} G_{\ell q,\ell r} z_q \bar{z}_r\right] - k^{-1}\frac{\Delta^2}{2}\left[\sum_{|\alpha|,|\beta|=2} \frac{G_{\alpha,\beta}}{\alpha!\beta!} z^\alpha \bar{z}^\beta\right] + \mathcal{O}\left(k^{-2}\right)\right) \\ &= \left(\frac{2\pi}{k}\right)^n \left(1 + k^{-1}\frac{1}{2}\sum_{\ell,q=1}^{n} G_{\ell q,\ell q} + \mathcal{O}\left(k^{-2}\right)\right).\end{aligned}$$

We get the conclusion by using Eq. (2) and the fact that $\|s_k\|^2 = \int_U |\psi|^2 e^{-k\varphi}\mu + \mathcal{O}(e^{-k/C})$. □

**Lemma 3.3.** *There exists a sequence $(C_k)$ of positive numbers such that for any $k \geqslant 1$, for any $s \in H^0(U, L^k)$, we have*

$$\left| s(p) \right|^2 \leqslant \left(\frac{k}{2\pi}\right)^n \left(1 + k^{-1}\frac{\rho(p)}{2} + k^{-3/2} C_k\right) \int_{B(\frac{\ln k}{\sqrt{k}})} |s|^2 \mu,$$

*where for any $r > 0$, $B(r) = \{q \in U / \sum |z_\ell(q)|^2 \leqslant r^2\}$. Furthermore, for any $\epsilon > 0$, $C_k = \mathcal{O}(k^\epsilon)$.*

**Proof.** Consider the polynomial $\varphi_4$ and $d_2$ introduced previously. Working in polar coordinates, one sees that the integral of $z^\alpha \bar{z}^\beta \mu_{\mathrm{Leb}}$ on $B(r)$ vanishes if $\alpha \neq \beta$. Consequently, for any non-vanishing $\alpha \in \mathbb{N}^n$, we have that

$$\int_{B(r)} \mathrm{e}^{-k\varphi_4} z^\alpha \, d_2 \mu_{\mathrm{Leb}} = 0.$$

So for any holomorphic function $f : U \to \mathbb{C}$,

$$\int_{B(r)} \mathrm{e}^{-k\varphi_4} |f|^2 \, d_2 \mu_{\mathrm{Leb}} = \int_{B(r)} \mathrm{e}^{-k\varphi_4} \big( \big| f(p) \big|^2 + \big| f - f(p) \big|^2 \big) \, d_2 \mu_{\mathrm{Leb}}.$$

We obtain that

$$\big| f(p) \big|^2 \leqslant \frac{\int_{B(r)} \mathrm{e}^{-k\varphi_4} |f|^2 \, d_2 \mu_{\mathrm{Leb}}}{\int_{B(r)} \mathrm{e}^{-k\varphi_4} \, d_2 \mu_{\mathrm{Leb}}}. \tag{3}$$

Since $\varphi = \varphi_4 + \mathcal{O}(|z|^5)$ and $\mu / \mu_{\mathrm{Leb}} = d_2 + \mathcal{O}(|z|^3)$, there exists $C > 0$ such that on a neighborhood of $p$:

$$-\varphi_4 \leqslant -\varphi + C|z|^5 \quad \text{and} \quad d_2 \leqslant \frac{\mu}{\mu_{\mathrm{Leb}}} \big(1 + C|z|^3\big).$$

So if $|z| \leqslant r/\sqrt{k}$, then

$$-k\varphi_4 \leqslant -k\varphi + Cr^5 k^{3/2} \quad \text{and} \quad d_2 \leqslant \frac{\mu}{\mu_{\mathrm{Leb}}} \big(1 + Cr^3 k^{-3/2}\big).$$

Consequently,

$$\int_{B(r/\sqrt{k})} \mathrm{e}^{-k\varphi_4} |f|^2 d_2 \mu_{\mathrm{Leb}} \leqslant \mathrm{e}^{Cr^5 k^{-3/2}} \big(1 + Cr^3 k^{-3/2}\big) \int_{B(r/\sqrt{k})} \mathrm{e}^{-k\varphi} |f|^2 \mu.$$

If $r = \ln k$, then

$$\mathrm{e}^{Cr^5 k^{-3/2}} \big(1 + Cr^3 k^{-3/2}\big) = 1 + \mathcal{O}\big((\ln k)^5 k^{-3/2}\big)$$

so that

$$\int_{B(\frac{\ln k}{\sqrt{k}})} \mathrm{e}^{-k\varphi_4} |f|^2 d_2 \mu_{\mathrm{Leb}} \leqslant \big(1 + \mathcal{O}\big((\ln k)^5 k^{-3/2}\big)\big) \int_{B(\frac{\ln k}{\sqrt{k}})} \mathrm{e}^{-k\varphi} |f|^2 \mu. \tag{4}$$

Applying the Laplace method as in the proof of Lemma 3.2, we have:

$$\int_{B(\frac{\ln k}{\sqrt{k}})} \mathrm{e}^{-k\varphi_4} d_2 \mu_{\mathrm{Leb}} = \left( \frac{2\pi}{k} \right)^n \left( 1 - k^{-1} \frac{\rho(p)}{2} + \mathcal{O}\big(k^{-2}\big) \right). \tag{5}$$

Gathering the estimates (3), (4) and (5), we obtain:

$$\big| f(p) \big|^2 \leqslant \left( \frac{k}{2\pi} \right)^n \left( 1 + k^{-1} \frac{\rho(p)}{2} + k^{-3/2} C_k \right) \int_{B(\frac{\ln k}{\sqrt{k}})} \mathrm{e}^{-k\varphi} |f|^2 \mu$$

where $C_k = \mathcal{O}((\ln k)^5)$. □

By the previous lemmas, we obtain:

$$\Pi_k(p, p) = \left( \frac{k}{2\pi} \right)^n \left( 1 + k^{-1} \frac{\rho(p)}{2} + \mathcal{O}\big(k^{-3/2+\epsilon}\big) \right)$$

for any $\epsilon > 0$. This concludes the proof.

As a final remark, let us point that the characterization in Lemma 3.1 is well known. The peaked section $s_k$ and its norm estimate, Lemma 3.2, were already in [15]. The original argument is the proof of the upper bound, Lemma 3.3, especially inequality (3).

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