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Algebraic geometry

The stability of Frobenius direct images of rank-two bundles over surfaces

La stabilité de l'image directe de Frobenius des fibrés de rang deux sur des surfaces

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A R T I C L E   I N F O

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A B S T R A C T

Let X be a smooth projective surface over an algebraically closed field k of characteristic $p \geq 5$ with Ω_X^1 semistable and $\mu(\Omega_X^1) > 0$. Given a semistable (resp. stable) vector bundle W of rank 2, we prove that the direct image F_*W under the Frobenius morphism F is also semistable (resp. stable).

R É S U M É

Soit X une surface projective lisse sur un corps algébriquement clos k de caractéristique $p \geq 5$ avec Ω_X^1 semistable et $\mu(\Omega_X^1) > 0$. Étant donné un fibré vectoriel semistable (resp. stable) W de rang 2 sur X, on montre que l'image directe F_*W par le morphisme de Frobenius F est aussi semistable (resp. stable).

1. Introduction

Let X be a smooth projective variety of dimension n over an algebraically closed field k with $\mathrm{char}(k) = p > 0$. The absolute Frobenius morphism $F_X : X \to X$ is induced by $\mathcal{O}_X \to \mathcal{O}_X$, $f \mapsto f^p$. Let $F : X \to X_1 := X \times_k k$ denote the relative Frobenius morphism over k. This endomorphism of X is of fundamental importance in algebraic geometry over characteristic $p > 0$. One of the themes is to study its action on the geometric objects on X.

It is well known that F_* preserves the stability of vector bundles on curves of genus $g \geq 2$ (see [2, p. 620, Theorem 3.3] (for low p), [6, p. 431, Theorem 2.2]). In higher dimensions, X. Sun [6, p. 447, Corollary 4.9] proves that the instability of F_*E is bounded by the instability of $E \otimes \mathrm{T}^\ell(\Omega_X^1)$ $(0 \leq \ell \leq n(p-1))$ for any vector bundle E. Especially for the surface, X. Sun also proves that the direct image $F_*\mathcal{L}$ is stable for a line bundle $\mathcal{L}$ with Ω_X^1 semistable and $\mu(\Omega_X^1) > 0$ (see [7, p. 176, Theorem 4.7]). But it is unknown whether F_* preserves the stability of a high-rank vector bundle over a smooth projective

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surface with Ω_X^1 semistable and $\mu(\Omega_X^1) > 0$. For a special kind of rank-two vector bundle, X. Sun has proved the following theorem, which is a slightly generalized version of [3, p. 245, Theorem 3.1].

Theorem 1. *(See [7, p. 179, Theorem 4.9].) Let X be a smooth projective surface with $\mu(\Omega_X^1) > 0$. Assume that Ω_X^1 is semistable. Then $F_*(\mathcal{L} \otimes \Omega_X^1)$ is semistable for any line bundle $\mathcal{L}$ on X. Moreover, if Ω_X^1 is stable, then $F_*(\mathcal{L} \otimes \Omega_X^1)$ is stable.*

In this note, we generalize the above theorem to an arbitrary rank-two semistable vector bundle W with some restriction on the characteristic p as following:

Theorem 2. *Let X be a smooth projective surface over an algebraically closed field k of characteristic $p \geq 5$ with Ω_X^1 semistable and $\mu(\Omega_X^1) > 0$. Then F_*W is semistable (resp. stable) for any rank-two semistable (resp. stable) vector bundle W.*

Remark 1. (1) Assume that Ω_X^1 is semistable and $\mu(\Omega_X^1) = 0$. Then all semistable vector bundles on X are strongly semistable (see [7, p. 170, Corollary 3.10]). In this case, we have that F_*E is strongly semistable for a semistable vector bundle E with any rank (see [5, p. 186, Proposition 4.4]).

(2) Let $\mathcal{J}$ be a smooth elliptic curve, and X be the fiber product $\mathcal{J} \times \mathcal{J}$. Then it is easy to check that $\Omega_X^1 = \mathcal{O}_X \oplus \mathcal{O}_X$. Thus X is smooth projective surface with Ω_X^1 is semistable and $\mu(\Omega_X^1) = 0$. According to a direct computation, we have $\mu(F_*\mathcal{O}_X) = 0$ by using Riemann–Roch Theorem. But the structure morphism $\mathcal{O}_X \to F_*\mathcal{O}_X$ implies that $F_*\mathcal{O}_X$ has a subsheaf with slope 0; consequently, $F_*\mathcal{O}_X$ is not stable. Thus we cannot relax the assumption $\mu(\Omega_X^1) > 0$ to $\mu(\Omega_X^1) \geq 0$ in order to guarantee that F_* preserves the stability of the vector bundles in the above two theorems.

2. Proof of Theorem 2

Let X be a smooth projective surface over k and $F : X \to X_1$ be the relative Frobenius morphism. Let us fix an ample divisor H. For a torsion-free sheaf $\mathcal{E}$ on X, we define the slope of $\mathcal{E}$ by:

$$\mu(\mathcal{E}) = \frac{c_1(\mathcal{E}) \cdot H}{\mathrm{rk}(\mathcal{E})}.$$

Definition 1. A torsion-free sheaf $\mathcal{E}$ on X is called semistable (resp. stable) if for any $0 \neq \mathcal{E}' \subset \mathcal{E}$, we have:

$$\mu(\mathcal{E}') \leq \mu(\mathcal{E}) \quad (\mathit{resp.}\ \mu(\mathcal{E}') < \mu(\mathcal{E})).$$

Moreover, a sheaf $\mathcal{E}$ is called strongly semistable (resp. stable) if its pullback by the k-th power F^k of Frobenius is semistable (resp. stable) for any $k \geq 0$.

For any torsion-free sheaf $\mathcal{E}$ on X, there is a unique filtration, the so-called Harder–Narasimhan filtration $\mathrm{HN}_\bullet(\mathcal{E})$ (see [1, p. 16, Definition 1.3.2]):

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_k = \mathcal{E}$$

such that $\mathrm{gr}_i^{\mathrm{HN}}(\mathcal{E}) = \mathcal{E}_i/\mathcal{E}_{i-1}$ $(1 \leq i \leq k)$ are semistable torsion free sheaves and

$$\mu(\mathcal{E}_1) > \mu(\mathcal{E}_2/\mathcal{E}_1) > \cdots > \mu(\mathcal{E}_k/\mathcal{E}_{k-1}).$$

We write

$$\mu_{\max}(\mathcal{E}) = \mu(\mathcal{E}_1); \qquad \mu_{\min}(\mathcal{E}) = \mu(\mathcal{E}_k/\mathcal{E}_{k-1})$$

and the instability of $\mathcal{E}$ is defined as

$$\mathrm{I}(\mathcal{E}) = \mu_{\max}(\mathcal{E}) - \mu_{\min}(\mathcal{E}).$$

Then it is easy to see that for any subbundle $\mathcal{E}' \subset \mathcal{E}$ we have:

$$\mu(\mathcal{E}') - \mu(\mathcal{E}) \leq \mathrm{I}(\mathcal{E}) = \mu_{\max}(\mathcal{E}) - \mu_{\min}(\mathcal{E})$$

and $\mathcal{E}$ is semistable if and only if $\mathrm{I}(\mathcal{E}) = 0$. Set

$$L_{\max}(\mathcal{E}) = \lim_{k \to \infty} \frac{\mu_{\max}((F^k)^*\mathcal{E})}{p^k}.$$

Note that the sequence $\frac{\mu_{\max}((F^k)^*\mathcal{E})}{p^k}$ is weakly increasing, so its limit exists (the limit may be infinite). One can see [4, p. 258] for details. The following three results will be used in the proof of Theorem 2.

Lemma 1. *(See [4, p. 275, Corollary 6.3].) If $L_{\max}(\Omega^1_X) \geq 0$, then*

$$L_{\max}\big(\Omega^1_X\big) \leq \frac{p}{p-1}\mu_{\max}\big(\Omega^1_X\big).$$

Lemma 2. *(See [5, p. 184, Proposition 4.2].) Let $\mathcal{E}_i$ $(1 \leq i \leq m)$ be torsion free sheaves. If $\mu(\Omega^1_X) \geq 0$, then*

$$\mathrm{I}\left(\bigotimes_{i=1}^{m}\mathcal{E}_i\right) \leq \frac{1}{p}\left(\sum_{i=1}^{m}\mathrm{rk}(\mathcal{E}_i) - m\right) \cdot \max\big\{0, L_{\max}\big(\Omega^1_X\big)\big\} + \sum_{i=1}^{m}\mathrm{I}(\mathcal{E}_i).$$

Lemma 3. *(See [7, p. 169, Proposition 3.9].) Let $\mathcal{E}$ be a semistable bundle. If all $\mathrm{gr}_i^{\mathrm{HN}}(F^*\mathcal{E})$ are strongly semistable, then*

$$\mathrm{I}\big(F^*\mathcal{E}\big) \leq \ell \cdot \mu_{\max}\big(\Omega^1_X\big),$$

where ℓ is the length of the Harder–Narasimhan filtration $\mathrm{HN}_\bullet(F^*\mathcal{E})$.

Now, we introduce the canonical filtration of $F^*(F_*W)$ for any vector bundle W, which has been fully studied in [2, p. 626, Section 5] and [6, p. 432, Section 3].

Definition 2. Let $V_0 := V = F^*(F_*W)$, $V_1 = \ker(F^*(F_*W) \twoheadrightarrow W)$ and

$$V_{\ell+1} = \ker\big(V_\ell \xrightarrow{\nabla} V \otimes_{\mathcal{O}_X} \Omega^1_X \to (V/V_\ell) \otimes_{\mathcal{O}_X} \Omega^1_X\big),$$

where $\nabla : V \to V \otimes_{\mathcal{O}_X} \Omega^1_X$ is the canonical connection.

The following theorem is a special case of [6, p. 437, Theorem 3.7 and p. 438, Corollary 3.8] for surfaces.

Theorem 3. *Let X be a smooth projective surface over k, then the filtration defined above is*

$$0 = V_{2(p-1)+1} \subset V_{2(p-1)} \subset \cdots \subset V_1 \subset V_0 = V = F^*(F_*W) \tag{1}$$

which has the following properties

(i) $\nabla(V_{\ell+1}) \subset V_\ell \otimes \Omega^1_X$ *for* $\ell \geq 1$, *and* $V_0/V_1 \cong W$.
(ii) $V_\ell/V_{\ell+1} \xrightarrow{\nabla} (V_{\ell-1}/V_\ell) \otimes \Omega^1_X$ *are injective morphisms of vector bundles for* $1 \leq \ell \leq 2(p-1)$, *which induce isomorphisms* $V_\ell/V_{\ell+1} = W \otimes \mathrm{T}^\ell\Omega^1_X$, *where*

$$\mathrm{T}^\ell\big(\Omega^1_X\big) = \begin{cases} \mathrm{Sym}^\ell(\Omega^1_X) & \text{when } \ell < p, \\ \mathrm{Sym}^{2(p-1)-\ell}(\Omega^1_X) \otimes \omega_X^{\ell-(p-1)} & \text{when } \ell \geq p. \end{cases}$$

In the case of characteristic zero, we know that the symmetric product and the tensor product of semistable bundles are also semistable. But in positive characteristic, these properties may not be true. However, by a theorem of Ilangovan–Mehta–Parameswaran (see [4, p. 274, Section 6]): If W_1, W_2 are semistable bundles with $\mathrm{rk}(W_1) + \mathrm{rk}(W_2) \leq p+1$, then $E_1 \otimes E_2$ is semistable. We have the following lemma which is implicit in [6, p. 179, Theorem 4.9] for the symmetric product of semistable bundles.

Lemma 4. *Let $\mathcal{F}$ be a semistable bundle on X of $\mathrm{rk}(\mathcal{F}) = 2$, then $\mathrm{Sym}^i\mathcal{F}$ is semistable when $i < p$.*

Proof. Consider the exact sequence

$$0 \to \mathrm{Sym}^{\ell-1}\mathcal{F} \otimes \det\mathcal{F} \to \mathrm{Sym}^\ell\mathcal{F} \otimes \mathcal{F} \to \mathrm{Sym}^{\ell+1}\mathcal{F} \to 0,$$

where all of the bundles have the same slope $(\ell+1) \cdot \mu(\mathcal{F})$ for all $\ell \geq 1$. For $i = 0$, 1, the result is trivial, and we prove the result by induction on i. Suppose $\mathrm{Sym}^i\mathcal{F}$ is semistable for $i \leq p-2$. Since

$$\mathrm{rk}\big(\mathrm{Sym}^i\mathcal{F}\big) + \mathrm{rk}(\mathcal{F}) = i + 3 \leq p + 1,$$

we have $\mathrm{Sym}^i\mathcal{F} \otimes \mathcal{F}$ is semistable. Consequently, $\mathrm{Sym}^{i+1}\mathcal{F}$ is semistable by the above exact sequence. □

Let $\mathcal{E} \subset F_*W$ be a nontrivial subsheaf, the canonical filtration (1) induces the filtration (we assume $V_m \cap F^*\mathcal{E} \neq 0$)

$$0 \subset V_m \cap F^*\mathcal{E} \subset \ldots \subset V_1 \cap F^*\mathcal{E} \subset V_0 \cap F^*\mathcal{E} = F^*\mathcal{E}.$$

Let

$$\mathcal{F}_\ell := \frac{V_\ell \cap F^*\mathcal{E}}{V_{\ell+1}\cap F^*\mathcal{E}} \subset \frac{V_\ell}{V_{\ell+1}}, \qquad r_\ell = \mathrm{rk}(\mathcal{F}_\ell).$$

Then $\mu(F^*\mathcal{E}) = \frac{1}{\mathrm{rk}(\mathcal{E})}\sum_{\ell=0}^m r_\ell \cdot \mu(\mathcal{F}_\ell)$ and

$$\mu(\mathcal{E}) - \mu(F_*W) = \frac{1}{p\cdot \mathrm{rk}(\mathcal{E})}\sum_{\ell=0}^m r_\ell\big(\mu(\mathcal{F}_\ell) - \mu\big(F^*F_*W\big)\big). \tag{2}$$

The following lemma is implicit in [7, p. 179].

Lemma 5. *Keep the above notations. If $m \neq 2(p-1)$,*

$$\mu(\mathcal{E}) - \mu(F_*W) \le \sum_{\ell=0}^m r_\ell \frac{\mathrm{I}(W\otimes \mathrm{T}^\ell(\Omega_X^1))}{p\cdot \mathrm{rk}(\mathcal{E})} - \frac{\mu(\Omega_X^1)}{p\cdot\mathrm{rk}(\mathcal{E})}(p-1); \tag{3}$$

if $m = 2(p-1)$, then there exists a nontrivial subsheaf $W' \subset W$ such that

$$\mu(\mathcal{E}) - \mu(F_*W) \le \sum_{\ell=0}^{2(p-1)} r'_\ell \cdot \frac{\mathrm{I}(W/W'\otimes \mathrm{T}^\ell(\Omega_X^1))}{p\cdot\mathrm{rk}(\mathcal{E})} + \frac{r_{2(p-1)}(\mathrm{rk}(F_*W) - \mathrm{rk}(\mathcal{E}))}{p\cdot\mathrm{rk}(\mathcal{E})\cdot\mathrm{rk}(W)}\big(\mu\big(W'\big) - \mu\big(W/W'\big)\big). \tag{4}$$

Proof. Combining Formula (2) and [7, p. 175, Lemma 4.5] which says that

$$\mu\big(F^*F_*W\big) = p\cdot\mu(F_*W) = \frac{p-1}{2}K_X\cdot H + \mu(W),$$

$$\mu(V_\ell/V_{\ell+1}) = \mu\big(W\otimes \mathrm{T}^\ell\big(\Omega_X^1\big)\big) = \frac{\ell}{2}K_X\cdot H + \mu(W),$$

we have

$$\mu(\mathcal{E}) - \mu(F_*W) = \sum_{\ell=0}^m r_\ell \frac{\mu(\mathcal{F}_\ell) - \mu(\frac{V_\ell}{V_{\ell+1}})}{p\cdot\mathrm{rk}(\mathcal{E})} - \frac{\mu(\Omega_X^1)}{p\cdot\mathrm{rk}(\mathcal{E})}\sum_{\ell=0}^m (p-1-\ell)r_\ell. \tag{5}$$

When $m \le p-1$, then (4.11) of [7, p. 175] implies that

$$\sum_{\ell=0}^m (p-1-\ell)r_\ell \ge (p-1).$$

When $p-1 < m < 2(p-1)$, let us recall (4.12) of [7, p. 175]:

$$\sum_{\ell=0}^m\left(\frac{n(p-1)}{2} - \ell\right)r_\ell = \sum_{\ell=m+1}^{n(p-1)}\left(\ell - \frac{n(p-1)}{2}\right)r_{n(p-1)-\ell} + \sum_{\ell > \frac{n(p-1)}{2}}^{m}\left(\ell - \frac{n(p-1)}{2}\right)(r_{n(p-1)-\ell} - r_\ell),$$

and $n = 2$ here. Then the facts $r_0 \ge 1$ and

$$r_{2(p-1)-\ell} - r_\ell \ge 0 \quad (\ell \ge p-1)$$

which is proved in the proof of [6, p. 446, Theorem 4.8] guarantee that

$$\sum_{\ell=0}^m (p-1-\ell)r_\ell \ge (p-1).$$

Above all, we have proved the first part of the result.

If $m = 2(p-1)$, then there exists a subsheaf $W' \subset W$ of rank $r_{2(p-1)}$ such that $\mathcal{F}_{2(p-1)} \cong W'\otimes \mathrm{T}^{2(p-1)}(\Omega_X^1)$ and $W'\otimes\mathrm{T}^\ell(\Omega_X^1) \hookrightarrow \mathcal{F}_\ell$ (see [7, p. 179]). Let

$$0 \to W'\otimes\mathrm{T}^\ell\big(\Omega_X^1\big) \to \mathcal{F}_\ell \to \mathcal{F}'_\ell \to 0$$

be the induced exact sequence with $\mathcal{F}'_\ell \subset W/W'\otimes\mathrm{T}^\ell(\Omega_X^1)$. Then

$$\mu(\mathcal{F}_\ell) - \mu\left(\frac{V_\ell}{V_{\ell+1}}\right) \le \frac{r_{2(p-1)}(\mathrm{rk}(\frac{V_\ell}{V_{\ell+1}}) - r_\ell)}{r_\ell\cdot\mathrm{rk}(W)}\big(\mu\big(W'\big) - \mu\big(W/W'\big)\big) + \frac{r'_\ell}{r_\ell}\cdot\mathrm{I}\big(W/W'\otimes\mathrm{T}^\ell\big(\Omega_X^1\big)\big)$$

where $r'_\ell = \mathrm{rk}(\mathcal{F}'_\ell)$. Substituting it into Formula (5) and noting that (see [7, p. 175, Lemma 4.6])

$$\sum_{\ell=0}^{m}(p-1-\ell)r_\ell \geq 0,$$

we have the required result. □

Now, we prove Theorem 2.

Proof of Theorem 2. We first assume that W is a semistable bundle of rank 2, and prove that F_*W is semistable. By Lemma 4,

$$\mathrm{T}^\ell(\Omega^1_X) = \begin{cases} \mathrm{Sym}^\ell(\Omega^1_X) & \text{when } \ell < p, \\ \mathrm{Sym}^{2(p-1)-\ell}(\Omega^1_X) \otimes \omega_X^{\ell-(p-1)} & \text{when } \ell \geq p \end{cases}$$

are semistable, where $\omega_X = \mathcal{O}_X(K_X)$ is the canonical line bundle of X.

Keep the notations as above. If $m = 2(p-1)$, then

$$\mu(\mathcal{E}) - \mu(F_*W) \leq 0$$

is a direct corollary of Lemma 5.

If $m \neq 2(p-1)$, we have

$$\mu(\mathcal{E}) - \mu(F_*W) \leq \sum_{\ell=0}^{m} r_\ell \frac{\mu(\mathcal{F}_\ell) - \mu(\frac{V_\ell}{V_{\ell+1}})}{p \cdot \mathrm{rk}(\mathcal{E})} - \frac{\mu(\Omega^1_X)}{p \cdot \mathrm{rk}(\mathcal{E})} \cdot (p-1).$$

It can be easily checked that $V_\ell/V_{\ell+1} = W \otimes \mathrm{T}^\ell(\Omega^1_X)$ are semistable, except that

$$V_{p-1}/V_p = W \otimes \mathrm{Sym}^{p-1}\big(\Omega^1_X\big)$$

may not be semistable. Thus

$$\mu(\mathcal{E}) - \mu(F_*W) \leq r_{p-1} \frac{\mu(\mathcal{F}_{p-1}) - \mu(\frac{V_{p-1}}{V_p})}{p \cdot \mathrm{rk}(\mathcal{E})} - \frac{\mu(\Omega^1_X)}{p \cdot \mathrm{rk}(\mathcal{E})} \cdot (p-1).$$

If $r_{p-1} = 0$, there is nothing to prove. If $r_{p-1} > 0$, we will prove

$$r_{p-1} \cdot \big(\mu(\mathcal{F}_{p-1}) - \mu(V_{p-1}/V_p)\big) \leq \mu\big(\Omega^1_X\big)(p-1)$$

by using $p \geq 5$. Consider the following two exact sequences

$$0 \to \mathrm{Sym}^{p-2}\big(\Omega^1_X\big) \otimes \omega_X \otimes W \to V_{p-1}/V_p \otimes \Omega^1_X \to \mathrm{Sym}^p\big(\Omega^1_X\big) \otimes W \to 0, \tag{6}$$

$$0 \to F^*\Omega^1_X \otimes W \to \mathrm{Sym}^p\big(\Omega^1_X\big) \otimes W \to \mathrm{Sym}^{p-2}\big(\Omega^1_X\big) \otimes \omega_X \otimes W \to 0 \tag{7}$$

where all of the bundles have the same slope $p \cdot \mu(\Omega^1_X) + \mu(W)$. For $\mathcal{F}_{p-1} \subset V_{p-1}/V_p$, we can obtain an exact sequence

$$0 \to \mathcal{F}'_{p-1} \to \mathcal{F}_{p-1} \otimes \Omega^1_X \to \mathcal{F}''_{p-1} \to 0$$

from the above exact sequence (6), where

$$\mathcal{F}'_{p-1} \subset \mathrm{Sym}^{p-2}\big(\Omega^1_X\big) \otimes \omega_X \otimes W, \qquad \mathcal{F}''_{p-1} \subset \mathrm{Sym}^p\big(\Omega^1_X\big) \otimes W.$$

If $\mathcal{F}''_{p-1}$ is trivial, then we are done since $\mathrm{Sym}^{p-2}(\Omega^1_X) \otimes \omega_X \otimes W$ is semi-stable with slope $\mu(V_{p-1}/V_p) + \mu(\Omega^1_X)$. If not, we obtain the following inequality:

$$2r_{p-1} \cdot \big(\mu(\mathcal{F}_{p-1}) - \mu(V_{p-1}/V_p)\big) \leq \mathrm{rk}\big(\mathcal{F}''_{p-1}\big) \cdot \big(\mu\big(\mathcal{F}''_{p-1}\big) - \mu(V_{p-1}/V_p) - \mu\big(\Omega^1_X\big)\big)$$

by putting the inequality

$$\mu\big(\mathcal{F}'_{p-1}\big) \leq \mu\big(\mathrm{Sym}^{p-2}\big(\Omega^1_X\big) \otimes \omega_X \otimes W\big) = \mu(V_{p-1}/V_p) + \mu\big(\Omega^1_X\big)$$

into the equality

$$\mu\big(\mathcal{F}_{p-1} \otimes \Omega^1_X\big) = \frac{\mathrm{rk}(\mathcal{F}'_{p-1})}{2r_{p-1}} \mu\big(\mathcal{F}'_{p-1}\big) + \frac{\mathrm{rk}(\mathcal{F}''{}_{p-1})}{2r_{p-1}} \mu\big(\mathcal{F}''_{p-1}\big).$$

Thus it is enough to show

$$\mathrm{rk}(\mathcal{F}''_{p-1})\cdot\big(\mu(\mathcal{F}''_{p-1})-\mu(V_{p-1}/V_p)-\mu(\Omega^1_X)\big)\le 2(p-1)\mu(\Omega^1_X).$$

The exact sequence (7) induces an exact sequence

$$0\to E_1\to\mathcal{F}''_{p-1}\to E_2\to 0$$

where $E_1\subset W\otimes F^*\Omega^1_X$, $E_2\subset \mathrm{Sym}^{p-2}(\Omega^1_X)\otimes\omega_X\otimes W$. If $E_1=0$, it is clear for $\mathrm{Sym}^{p-2}(\Omega^1_X)\otimes\omega_X\otimes W$ is semi-stable of slope $\mu(V_{p-1}/V_p)+\mu(\Omega^1_X)$. If $E_1\neq 0$, we have

$$\mathrm{rk}(\mathcal{F}''_{p-1})\cdot\big(\mu(\mathcal{F}''_{p-1})-\mu(V_{p-1}/V_p)-\mu(\Omega^1_X)\big)\le \mathrm{rk}(E_1)\big(\mu(E_1)-\mu(W\otimes F^*\Omega^1_X)\big)$$

by using the same argument as above. If $\mathrm{rk}(E_1)=4$, then $E_1=W\otimes F^*\Omega^1_X$ and we have:

$$\mathrm{rk}(E_1)\big(\mu(E_1)-\mu(W\otimes F^*\Omega^1_X)\big)=0.$$

Else $\mathrm{rk}(E_1)\le 3$, let us consider the instability $\mathrm{I}(W\otimes F^*\Omega^1_X)$. By using Lemma 1 and Lemma 2, we have:

$$\begin{aligned}\mathrm{I}\big(W\otimes F^*\Omega^1_X\big)&\le\frac{\mathrm{rk}(W)+\mathrm{rk}(F^*\Omega^1_X)-2}{p}\cdot\max\big\{0,L_{\max}(\Omega^1_X)\big\}+\mathrm{I}\big(F^*\Omega^1_X\big)+\mathrm{I}(W)\\&\le\frac{2}{p-1}\cdot\mu(\Omega^1_X)+\mathrm{I}\big(F^*\Omega^1_X\big).\end{aligned}$$

If $F^*\Omega^1_X$ is semistable, then $\mathrm{I}(F^*\Omega^1_X)=0$. Otherwise we have $\mathrm{I}(F^*\Omega^1_X)\le 2\mu(\Omega^1_X)$ by using Lemma 3. From $p\ge 5$, we obtain:

$$\mathrm{rk}(E_1)\big(\mu(E_1)-\mu(W\otimes F^*\Omega^1_X)\big)\le\frac{6}{p-1}\mu(\Omega^1_X)+6\mu(\Omega^1_X)\le 2(p-1)\mu(\Omega^1_X).\tag{8}$$

To sum up, one has

$$\mu(\mathcal{E})-\mu(F_*W)\le 0$$

which implies that F_*W is semistable.

If W is stable, we can prove the stability of F_*W similarly. If $m=2(p-1)$, then formula (4) in Lemma 5 implies that

$$\mu(\mathcal{E})-\mu(F_*W)<0$$

from $\mu(W')<\mu(W/W')$, which is a direct corollary of the stability of W. When $m\neq 2(p-1)$, if $r_{p-1}=0$, there is nothing to prove. If $r_{p-1}\neq 0$, repeat the above process and note that

$$\mathrm{rk}(E_1)\big(\mu(E_1)-\mu(W\otimes F^*\Omega^1_X)\big)\le\frac{6}{p-1}\mu(\Omega^1_X)+6\mu(\Omega^1_X)$$

in the inequality (8), we obtain that

$$\begin{aligned}\mu(\mathcal{E})-\mu(F_*W)&\le r_{p-1}\frac{\mu(\mathcal{F}_{p-1})-\mu(\frac{V_{p-1}}{V_p})}{p\cdot\mathrm{rk}(\mathcal{E})}-\frac{\mu(\Omega^1_X)}{p\cdot\mathrm{rk}(\mathcal{E})}\cdot(p-1)\\&\le\frac{4-p+\frac{3}{p-1}}{p\cdot\mathrm{rk}(\mathcal{E})}\mu(\Omega^1_X)<0,\end{aligned}$$

where the last inequality comes from the assumption $p\ge 5$. □

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