Geometry/Algebraic geometry

A characterization of *d*-uple Veronese varieties

Une caractérisation des variétés d-uples de Veronese

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A R T I C L E   I N F O

A B S T R A C T

We characterize *d*-uple Veronese embeddings of finite-dimensional projective spaces. The easiest non-trivial instance of our theorem is the embedding of the projective plane in a 5-dimensional projective space, a result obtained in 1901 by Severi when the underlying field is the field of complex numbers.

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R É S U M É

Nous caractérisons les plongements *d*-uples de Veronese d'espaces projectifs de dimension finie. L'instance non triviale la plus simple de notre théorème est le plongement du plan projectif dans un espace projectif de dimension 5, un résultat obtenu en 1901 par Severi lorsque le corps sous-jacent est le corps des nombres complexes.

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1. Introduction

In 1901, Severi [17] characterized the Veronese surface as the only smooth irreducible and non-degenerate projective surface in a 5-dimensional complex projective space that can be projected isomorphically into a 4-dimensional complex projective space. In 1921, this theorem was strengthened by C. Segre [15], see [10] for a modern account.

In 1984, Mazzocca and Melone [9] used three simple axioms (the MM-axioms) to characterize the ordinary quadric Veronese variety over finite fields of odd order. This was extended to all finite fields by Hirschfeld and Thas [5] and to arbitrary fields by Schillewaert and Van Maldeghem [14]. Our main result holds for an arbitrary finite degree d and arbitrary fields of cardinality at least $\frac{(d+1)^2}{2}$ using a generalization of the MM-axioms. As Severi's conditions imply the MM-axioms [7,21] for the case of surfaces in a 5-dimensional space, Severi's theorem is a particular instance of our result.

The *Veronese embedding* $\phi_{n,d} : \mathbb{P}^n(\mathbb{K}) \to \mathbb{P}^{\binom{n+d}{d}-1}(\mathbb{K})$ maps $(x_0, \ldots, x_n)$ to $(x_0^d, x_0^{d-1}x_1, \ldots, x_n^d)$. The *Veronese variety* $\mathcal{V}_n^{d^n}$ is the image of this map. A *rational normal curve C* in a d-dimensional projective space Σ is a set of points of Σ that is

projectively equivalent to $\mathcal{V}_1^d$. For any $x \in C$ one can define a unique *tangent line* $T_x(C)$, which is determined by the points of the curve if either $d = 2$ or $|\mathbb{K}| \geq d + 2$.

Let V be a possibly infinite-dimensional, non-trivial vector space over some field $\mathbb{K}$, and let $\mathbb{P}(V)$ be the corresponding projective space. Let X be a spanning point set of $\mathbb{P}(V)$ and let $\varXi$ be a collection of d-dimensional subspaces of $\mathbb{P}(V)$, the *rational spaces* of X, such that, for any $\xi \in \varXi$, the intersection $\xi \cap X$ is a rational normal curve $X(\xi)$ in ξ of dimension d ($d \geq 2$) and then, for $x \in X(\xi)$, we sometimes denote $T_x(X(\xi))$ simply by $T_x(\xi)$. We call $\mathcal{V} = (X, \varXi)$, or briefly X, a *Veronesean cap of degree d* if the following properties (V1), (V2) and (V3) hold.

(V1) Any two points x and y of X are contained in an element of $\varXi$, denoted by $[x, y]$ if unique.
(V2) If $\xi_1, \xi_2 \in \varXi$, with $\xi_1 \neq \xi_2$, then $\xi_1 \cap \xi_2 \subset X$.
(V3) For every $x \in X$, every $\xi \in \varXi$, with $x \notin \xi$, $\bigcup_{y \in X(\xi)} T_x([x, y])$ is a plane $T(x, \xi)$ depending on x and ξ.

The condition (V1) alone is not sufficient to characterize Veronese varieties, as for instance geometries related to the non-split version of the second row of the Freudenthal–Tits Magic square, e.g., coming from the real form $\mathsf{E}_{6,2}^{28}$ of a group of exceptional type E_6 satisfy (V1).

Projecting the Veronese variety $\mathcal{V}_2^{d^2}$ from a point y contained in the span of two of its rational subspaces, but not contained in the span of a point of the variety and a rational space, nor in any of the planes $T(x, \xi)$, yields an example that satisfies (V1) and (V3), but not (V2). (Such a point y exists for finite fields $\mathbb{K}$ and large enough d by counting.) Moreover, there are examples satisfying (V1) and (V2), but not (V3), e.g., Veroneseans of Hermitian unitals.

In the complex case, every (possibly singular) non-degenerate curve in $\mathbb{P}^d$ of degree d is a rational normal curve [18, p. 121]. In the finite case, one can sometimes relax the conditions [2], but not always [3,16].

A Veronese variety is a Veronesean cap: By Lemma 2.3 of [6] the images of lines are rational normal curves. Property (V1) is immediate. To verify (V2), by homogeneity, see [6], we only need to check two situations, namely $\psi_{1,2} := \langle \phi_{n,d}(L_1) \rangle \cap \langle \phi_{n,d}(L_2) \rangle$, and if $n \geq 3$ we need to compute also $\psi_{1,3} := \langle \phi_{n,d}(L_1) \rangle \cap \langle \phi_{n,d}(L_3) \rangle$, where L_1 is given by $X_i = 0$, $2 \leq i \leq N$, L_2 by $X_i = 0$, $i \neq 2, 1 \leq i \leq N$ and L_3 by $X_i = 0$; $0 \leq i \leq N - 2$. Then a computation yields $\psi_{1,2} := \phi_{n,d}(L_1 \cap L_2)$ and $\psi_{1,3} = \emptyset$. For (V3), again by homogeneity, consider the point $r := (0, \ldots, 0, 1)$ and the points $p_1 := (1, 0, \ldots, 0)$ and $p_2 := (0, 1, 0, \ldots, 0)$ on L_1. Then a computation yields that the plane spanned by the tangent lines at $\phi_{n,d}(r)$ to $\phi_{n,d}(\langle r, p_1 \rangle)$ and $\phi_{n,d}(\langle r, p_2 \rangle)$ is the requested one.

2. Statement of the Main Result

Proposition 2.1. *If X is a Veronesean cap, then for every $x \in X$ the set $\bigcup_{\xi \in \varXi, x \in \xi} T_x(\xi)$ is a subspace of constant dimension.*

We call the dimension of these subspaces the *index*. In Proposition 4.4 we prove an upper bound on the dimension of a Veronesean cap of finite index.

Lemma 2.2. *If X is a Veronesean cap of degree d and finite index $n \geq 2$ in a projective space $\mathbb{P}(V)$ over a field $\mathbb{K}$, then $\dim V \leq M$.*

We classify *extremal* Veronesean caps, i.e. those of largest possible dimension with respect to their degree and index, over fields of sufficiently large cardinality with respect to the degree.

Main Result. *If X is a Veronesean cap of degree d and finite index $n \geq 2$ in a projective space $\mathbb{P}(V)$ over a field $\mathbb{K}$ where $\dim V = M := \binom{n+d}{d}$ and such that $|\mathbb{K}| \geq \frac{(d+1)^2}{2}$, then X is projectively equivalent to the Veronese variety $\mathcal{V}_n^{d^n}$ over $\mathbb{K}$.*

In the complex case, a similar result was proved already in 1921 by Bompiani [1], extending C. Segre's result and more recently for arbitrary n by Pirio and Russo [12], Pirio and Trépreau [13]. They assume (V1) and a global version of (V3), but do not require (V2). On the other hand, they require X to be a variety from the outset. For the case $d = 2$, it is not difficult to deduce our axioms from Severi's, see, e.g., [7].

An interesting open problem is whether an analogue of Theorem 2.2 could be proved over arbitrary fields not using (V2) and using a global version of (V3). In particular, we are unaware of examples where (V1) and (V3) and $\dim V = M$ hold, but not (V2). Also the general study of projections of the Veronese variety in this axiomatic setting deserves further attention.

Remark 2.3. One can often relax the bound on the field size in the Main Result, as in fact three separate bounds appear in our proof. For $n = 2, d \geq 6$, we need the strongest bound of the three, which is $|\mathbb{K}| \geq \frac{(d+1)^2}{2}$, to assure the existence of an arc of size $d + 1$ in non-Desarguesian projective planes of order $|\mathbb{K}|$ by a result of Lunelli and Sce [8]. (Note that if $n \geq 3$ such an arc always exists.) A bound $|\mathbb{K}| \geq d + 2$ when $d \neq 2$ is necessary, as otherwise the tangent lines are not determined by the points of the curve. Lastly, the bound

$$\frac{|\mathbb{K}|^n - 1}{|\mathbb{K}| - 1} \geq 2 + \sum_{i=2}^{\min(n-1,d)} (-1)^i \binom{d+1}{i+1} \frac{|\mathbb{K}|^{n-i} - 1}{|\mathbb{K}| - 1}$$

is used to prove Lemma 4.3. This bound is always satisfied if $|\mathbb{K}| \geq d^{\frac{3}{2}}$ and becomes irrelevant in view of the $d+2$ bound for $n \leq 2$ or $d \leq 10$.

3. Basic properties of rational normal curves

Let C be a rational normal curve in a d-dimensional projective space Σ as defined in Section 1 (with $d \geq 2$).

Lemma 3.1. *(See [6, Theorem 1.1].) All $d+1$ points of C are linearly independent.*

Observation 3.2. *(See [4, Ex. 1.19].) The rational normal curve C comes equipped with a notion of crossratio on quadruples of points, such that the group $\mathrm{PGL}(2, \mathbb{K})$ acts three-transitively on C while preserving the crossratio.*

Lemma 3.3. *Let x be a point of the curve C, then a projection from x to a hyperplane not incident with it maps $C \setminus \{x\}$ and the tangent line through x bijectively to the points of a rational normal curve in a $(d-1)$-dimensional projective space. This projection preserves the crossratio on C.*

Proof. By Observation 3.2, we may fix $x = (1, 0, \ldots, 0)$ in Σ. This reduces the problem to an easy calculation. □

Repeated application of Lemma 3.3 shows the following.

Corollary 3.4. *Let $x_1, \ldots, x_i$ be i pairwise different points of C such that $1 \leq i \leq d-1$. Then a projection from the span of $x_1, \ldots, x_i$ to a complementary subspace maps $C \setminus \{x\}$ and the tangent lines through $x_1, \ldots, x_i$ bijectively to the points of a rational normal curve in a $(d-i)$-dimensional projective space. This projection preserves the crossratios C.*

Observation 3.5. *By considering 4 points of C and applying Corollary 3.4 to $d-1$ other points on the curve (which always exist when $|\mathbb{K}| \geq 4$ by the $d+2$ bound of Remark 2.3), one obtains that the crossratio mentioned in Observation 3.2 solely depends on the point set of C. For $|\mathbb{K}| = 2$ or 3 this is trivial as there is only one way to define a valid crossratio on 3 or 4 points.*

Using Corollary 3.4 considering a projection from $\langle C \cap L_1, \ldots, C \cap L_i \rangle$ yields Corollary 3.6.

Corollary 3.6. *Let $L_1, \ldots, L_i$ and $x_1, \ldots x_j$ be respectively i tangent lines on C and j points of C, none of which are equal or incident with each other. If $2i + j \leq d+1$, then the span of these lines and points is of dimension $2i + j - 1$.*

4. Proof of the Main Result

4.1. The projective space associated with a Veronesean cap

We use the same notation as in Section 1. Associated with $\mathcal{V}$ we can consider the geometry $\mathcal{P}$ having point set X and line set Ξ, endowed with the natural incidence. Then a proof similar to [19, Prop. 2.2] yields Proposition 4.1.

Proposition 4.1. *$\mathcal{P}$ is a projective space of dimension $n \geq 2$.*

Also the proof of Proposition 2.1 is essentially the same as that of [19, Prop. 2.4]. The obtained index is the same n as in Proposition 4.1.

From now on, we denote a point of $\mathcal{P}$ with a $\bar{\ }$, e.g. $\bar{x}$, and the corresponding point in $\mathbb{P}(V)$ without a $\bar{\ }$, e.g. x. Similarly, we denote subspaces of $\mathcal{P}$ with a $\bar{\ }$, and if $\bar{\pi}$ is a subspace of $\mathcal{P}$, then $\pi := \langle x \in X | \bar{x} \in \bar{\pi} \rangle$. The following easy observation will be crucial for induction arguments.

Lemma 4.2. *Let $\bar{\pi}$ be some non-empty subspace of $\mathcal{P}$, then (X_π, Ξ_π) with $X_\pi := \{x | \bar{x} \in \bar{\pi}\}$, and $\Xi_\pi := \{\xi \in \Xi | X(\xi) \subset X_\pi\}$, is a Veronesean cap of degree d inside π, which we call a subcap.*

4.2. Dimensional analysis

In the remainder of the proof, let (X, Ξ) be a Veronesean cap of degree d and index n, satisfying the restrictions listed in the Main Result. We assume as induction hypothesis that the Main Result holds for Veronesean caps of index up to $n-1$, note that the cases of index 0 and 1 are trivial.

Lemma 4.3. *Let $\bar{\pi}_1, \bar{\pi}_2, \ldots \bar{\pi}_{d+1}$ be a set of hyperplanes of $\mathcal{P}$ in general position. If $\bar{x}$ is a point of $\mathcal{P}$ not contained in any of these hyperplanes, then there exist at least two different lines through x such that each point on these lines is contained in at most two of the hyperplanes $\bar{\pi}_1, \bar{\pi}_2, \ldots \bar{\pi}_{d+1}$.*

Proof. This holds as the right-hand side of the second inequality of Remark 2.3 counts the number of intersection points of at least three of these subspaces (via the inclusion–exclusion principle) plus two, while the left-hand side of this inequality counts the number of lines through x. □

For the following proposition, we need $d + 1$ hyperplanes of $\mathcal{P}$ in general position. Note that if $\mathcal{P}$ is Moufang (which is automatically true for $n \geq 3$) and $|\mathbb{K}| \geq d$ there do exist hyperplanes $\bar{\pi}_1, \bar{\pi}_2, \ldots, \bar{\pi}_{d+1}$ in general position, for example the dual of the point set of a rational normal curve in $\mathcal{P}$. If $\mathcal{P}$ is a projective plane that is not Moufang, then [8] asserts the existence of $d + 1$ points in general position as soon as $\mathbb{K} \geq \frac{(d+1)^2}{2}$.

Proposition 4.4. *Let $\bar{\pi}_1, \bar{\pi}_2, \ldots, \bar{\pi}_i$ $(0 \leq i \leq d + 1)$ be a number of hyperplanes of $\mathcal{P}$ in general position that is extendable to a set of $d + 1$ hyperplanes in general position. Then*

(i) *If $\dim V = M$, then $\operatorname{codim}\langle\pi_1, \pi_2, \ldots, \pi_i\rangle = \binom{n+d-i}{d-i}$.*
(ii) *$\dim V \leq M$.*
(iii) *If $\dim V = M$ then $\pi_i \cap \langle\pi_1, \ldots, \pi_{i-1}\rangle = \langle\pi_i \cap \pi_1, \ldots, \pi_i \cap \pi_{i-1}\rangle$.*
(iv) *If $\dim V = M$ and the geometric dimension of $\bar{\pi}$ is m, then the geometric dimension of π is $\binom{m+d}{d} - 1$. In particular, (X_π, Ξ_π) is a Veronesean cap of degree d and index m satisfying the restrictions listed in the Main Result.*

Proof. First we prove $\operatorname{codim}\langle\pi_1, \pi_2, \ldots, \pi_i\rangle \leq \binom{n+d-i}{d-i}$ by induction on both n and $d - i$, making use of Lemma 4.2. The case $n = 1$ follows by Section 3, so assume $n \geq 2$.

First assume that $d - i = -1$. Let x be any point in $X \setminus (\bigcup_{j=1}^{d+1} \pi_j)$, and let $\xi \in \Xi$ be a rational normal space through x such that each $y \in X(\xi)$ is contained in at most two of the subspaces π_i (which exists by Lemma 4.3). For such a ξ, we have that if y is contained in some subspace π_i, then it is contained in $\langle\pi_1, \pi_2, \ldots \pi_i\rangle$. If it is contained in two of these subspaces, then also its tangent line is contained in $\langle\pi_1, \pi_2, \ldots \pi_{d+1}\rangle$. Hence $\langle\pi_1, \pi_2, \ldots \pi_{d+1}\rangle$ contains ξ, and by varying x all of X.

Secondly assume that $d - i \geq 0$. In this case we pick a hyperplane $\bar{\pi}_{i+1}$ of $\mathcal{P}$ such that $\bar{\pi}_1, \bar{\pi}_2, \ldots, \bar{\pi}_{i+1}$ are in general position. The codimension of the span $\langle\pi_1, \pi_2, \ldots, \pi_i\rangle$ in $\mathbb{P}(V)$ equals the sum of the codimension c_1 of $\langle\pi_1, \pi_2, \ldots, \pi_i\rangle$ in $\langle\pi_1, \pi_2, \ldots, \pi_{i+1}\rangle$ and the codimension c_2 of $\langle\pi_1, \pi_2, \ldots, \pi_{i+1}\rangle$ in $\mathbb{P}(V)$. Then c_1 is at most the codimension of $\langle\pi_1 \cap \pi_{i+1}, \ldots, \pi_i \cap \pi_{i+1}\rangle$ in π_{i+1}, which is bounded from above by $\binom{(n-1)+d-i}{d-i}$ by induction on n, and c_2 is bounded from above by $\binom{n+d-(i+1)}{d-(i+1)}$ by induction on $d - i$. The sum of both is at most $\binom{n+d-i}{d-i}$.

The bound for $i = 0$ yields $\dim V \leq M$ (proving (ii)). If $\dim V = M$, then all bounds above have to be sharp, which proves (i) and (iii). Repeated application of (i) with $i = 1$ proves (iv). □

4.3. The projective space $\mathcal{P}$ is isomorphic to $\mathbb{P}^n(\mathbb{K})$

As we know, the dimension of $\mathcal{P}$ from Proposition 4.1 one can use part (iv) of Proposition 4.4 to restrict to the case $n = 2$ for the remainder of Section 4.3. Remember from Observations 3.2 and 3.5 that the point sets of the rational normal curves $\xi \cap X$ with $\xi \in \Xi$ come equipped with a natural crossratio on quadruples. By construction of $\mathcal{P}$, this yields a crossratio on the point sets of lines of $\mathcal{P}$. The next step is now to define a crossratio on line pencils of $\mathcal{P}$.

Observation 4.5. *Let $\bar{x}$ be an arbitrary point of $\mathcal{P}$ and identify each line $\bar{\zeta}$ through $\bar{x}$ with the tangent line $T_x(\zeta)$. Hence, by (V3), we obtain a correspondence of lines through $\bar{x}$ with lines in the tangent plane on x. The latter, being a line pencil in the projective space $\mathbb{P}(V)$, comes with a natural notion of a crossratio, which via identification provides a crossratio on the line pencil through $\bar{x}$ in $\mathcal{P}$.*

The next lemma links these crossratios together.

Lemma 4.6. *The perspectivity from the line pencil through a point $\bar{x}$ of $\mathcal{P}$ to the points on a line $\bar{\xi}$ of $\mathcal{P}$ not containing $\bar{x}$ preserves the crossratios defined on these.*

Proof. Extend the line $\bar{\xi}$ to a set of lines $\{\bar{\xi}, \bar{\xi}_2, \ldots \bar{\xi}_{d+1}\}$ in general position, such that only $\bar{\xi}_{d+1}$ contains $\bar{x}$.

Parts (i) and (ii) of Proposition 4.4 imply that the codimension of $\langle\xi_2, \ldots, \xi_d\rangle$ is 3. Hence we may consider a projection τ of $\mathbb{P}(V)$ from $\langle\xi_2, \ldots, \xi_d\rangle$ on some two-dimensional subspace disjoint from it. Proposition 4.4 also implies that τ maps ξ on some line L (as $\langle\xi, \xi_2, \ldots, \xi_d\rangle$ is of codimension one) and that x is mapped to a point disjoint from L (by part (iii) applied to ξ_{d+1} and $\langle\xi, \xi_2, \ldots, \xi_d\rangle$).

The points of the curve $X(\xi)$ not in $\xi_2 \cup \cdots \cup \xi_d$ and the tangent lines on ξ through the remaining points on this curve project to pairwise distinct points of L while preserving the crossratios, by Corollary 3.4 and part (iii) of Proposition 4.4. This Corollary 3.4 also states that this projection preserves crossratios, hence we have connected the crossratio on $\xi \cap X$ and the crossratio on L.

Let ξ' be some rational normal space through x. From Corollary 3.6 and counting the number of points of $X \cap \xi'$ contained in any of the $\xi, \xi_2, \ldots \xi_{d-1}$, while accounting for the tangent lines of the points that are contained in two of these rational normal spaces (applying (V3)), it follows that the image of ξ' under τ is at most one-dimensional. As this image contains $\tau(x)$ and a point of L it has to be a line.

In the image of the projection we hence obtain a perspectivity between the line pencil through the point $\bar{x}$ of $\mathcal{P}$ and the points on $\bar{\xi}$, preserving the crossratios on both. $\square$

This has the following immediate consequence.

Corollary 4.7. *The induced action on the rational normal curve $X(\xi)$ of the projectivity group of $\mathcal{P}$ w.r.t. to a line $\bar{\xi}$ of $\mathcal{P}$ preserves the natural crossratio on it.*

Let G be the projectivity group of $\mathcal{P}$ w.r.t. to some line $\bar{\xi}$ of $\mathcal{P}$. Then $G \leq \mathrm{PGL}(2, \mathbb{K})$ as the latter consists of those permutations preserving the crossratio. As $\mathrm{PGL}(2, \mathbb{K})$ acts sharply 3-transitive while G is at least 3-transitive, one has $G = \mathrm{PGL}(2, \mathbb{K})$. Moreover, by von Staudt's theorem [20], this implies that the projective plane $\mathcal{P}$ is Pappian, hence $\mathbb{K}$ is a field and $\mathcal{P}$ is isomorphic to $\mathbb{P}^2(\mathbb{K})$.

Let us mention another corollary.

Corollary 4.8. *Let $\bar{x}$ and $\bar{\xi}$ be a point and line of $\mathcal{P}$ such that $\bar{x} \notin \bar{\xi}$. Then the map*

$$X(\xi) \to T(x, \xi) : y \mapsto T_x([x, y])$$

is completely determined by the curve $X(\xi)$, the point x, and the images of at least three points in $X(\xi)$ under this map.

Proof. By Lemma 4.6 this map preserves crossratios, so knowing the image of three points suffices to completely determine the map. $\square$

4.4. Mapping X onto a Veronese variety

Let Y represent the point set of the projective space $\mathcal{P}$. By construction of $\mathcal{P}$, we have a bijective map $\iota : Y \to X : \bar{x} \mapsto x$. If we set X' and Ξ' to be respectively the point set and the set of rational spaces of a Veronese variety $\mathcal{V}_n^{d^n}$ over the field $\mathbb{K}$ with ambient vector space $\mathbb{P}(V')$, then, by construction of the variety, we also have a bijective map $\iota' : Y \to X'$, i.e. the Veronese embedding.

Our goal is to show that the bijective map $\varphi := \iota' \circ \iota^{-1}$ from X to X' lifts uniquely to a collineation between the ambient projective spaces. We do this by induction on subcaps Z of increasing index m.

4.4.1. The case $m = 1$

Here we have to consider a map φ defined on a subcap Z of index 1, so on the points of a certain rational normal curve $X(\xi)$ (with $\xi \in \Xi$). We want to show that φ extends uniquely to a collineation defined on the subspace ξ of $\mathbb{P}(V)$. This is the case if and only if φ preserves the crossratios on the rational normal curve up to a possible field automorphism (see for example [4, Ex. 1.19]).

We may assume that the index of X is 2 (by using (iv) of Proposition 4.4), so that $\mathcal{P}$ is a projective plane. As the projectivity group determines crossratios up to a possible field automorphism σ, and by the fact that this group is a feature of $\mathcal{P}$ and does not depend on the actual structure of X and X', it is implied that φ preserves the crossratio on $X(\xi)$ up to a field automorphism σ.

Remark 4.9. Moreover, by Lemma 4.6, the field automorphism σ does not depend on the choice of the subcap Z.

4.4.2. The case $m \geq 2$

We may suppose without loss of generality that $Z = X$, so $m = n$. Fix $d + 1$ hyperplanes $\bar{\pi}_1, \bar{\pi}_2, \ldots, \bar{\pi}_{d+1}$ of $\mathcal{P}$ in general position, obtained by considering the dual of a rational normal curve in $\mathbb{P}^n(\mathbb{K})$. The spaces $\pi_1, \ldots, \pi_{d+1}$ span $\mathbb{P}(V)$ by Proposition 4.4.

The induction hypothesis allows us to define a collineation ρ mapping the subspace π_1 to the subspace $\varphi(\pi_1)$, in such a way that ρ extends φ on $X \cap \pi_1$. We now extend ρ recursively to a collineation agreeing with φ on $X \cap (\bigcup_{j=1}^{d+1} \pi_j)$. Suppose that we already have defined ρ on the span of π_1 up to π_i, agreeing with φ on $X \cap (\bigcup_{j=1}^{i} \pi_j)$ ($1 \leq i \leq d$). Then, in order to extend ρ to a collineation defined on the span up to π_1 up to π_{i+1}, it suffices to extend the restriction of ρ on $\pi_{i+1} \cap \langle \pi_1, \ldots, \pi_i \rangle$ to π_{i+1}. This is possible because of the induction hypothesis and (iii) and (iv) of Proposition 4.4 (and using Remark 4.9 in the case $m = 2$ to avoid having conflicting field automorphisms). Eventually, we obtain a collineation ρ defined on the entirety of $\mathbb{P}(V)$, agreeing with φ on $X \cap (\bigcup_{j=1}^{d+1} \pi_j)$.

Proposition 4.10. *The collineation ρ agrees with φ on the entirety of X.*

Proof. Let x be any point in $X \setminus \bigcup_{j=1}^{d+1} \pi_j$. It suffices to show that x is uniquely determined by the points $X \cap (\bigcup_{j=1}^{d+1} \pi_j)$.

By Lemma 4.3, we can find two lines $\bar{\bar{\xi}}_1$ and $\bar{\bar{\xi}}_2$ through $\bar{x}$ such that each point on these lines is contained in at most two of the hyperplanes $\bar{\pi}_1, \bar{\pi}_2, \ldots, \bar{\pi}_{d+1}$. If $d \leq 3$, we can pick these such that each point on these lines is contained in at most one such hyperplane.

Assume $d > 3$, and let y be a point on ξ_1 contained in two of the hyperplanes, we may suppose these are π_1 and π_{d+1}. By the bounds in Remark 2.3, we have $|\mathbb{K}| \geq d + 2$. Next we reconstruct the tangent line $L := T_y(\xi_1)$ from the points in $X \cap (\bigcup_{j=1}^{d+1} \pi_j)$. We already know the position of y and some plane in which L is contained by property (V3). If $n > 2$, we know at least two of these planes, which determines L uniquely, so we are left with the case that $n = 2$.

As $|\mathbb{K}| \geq d + 2$ and by the construction of the lines $\bar{\pi}_1, \bar{\pi}_2, \ldots, \bar{\pi}_{d+1}$, there exists some lines $\bar{\pi}_{d+2}$ and $\bar{\pi}_{d+3}$ such that $\bar{\pi}_1, \bar{\pi}_2, \ldots, \bar{\pi}_{d+3}$ are in general position. As $\bar{\pi}_{d+2}$ and $\bar{\pi}_{d+3}$ intersect the $\bar{\pi}_1, \bar{\pi}_2, \ldots, \bar{\pi}_{d+1}$ in $d + 1$ pairwise different points, the position of the subspaces π_{d+2} and π_{d+3} is uniquely determined. Also the unique intersection point of both, which we denote by z, is determined.

Applying part (iii) of Proposition 4.4 to π_1 and π_{d+2} yields that $y, z \notin \langle \pi_i | 2 \leq i \leq d \rangle$. Part (iii) of Proposition 4.4 applied to π_1 and $\pi_2, \ldots, \pi_{d+1}$ also yields $T_y(\pi_1) \not\subset \langle \pi_i | 2 \leq i \leq d \rangle$, in particular the projections of the tangent lines $T_y(\pi_1)$ and $T_y(\pi_{d+1})$, and hence all tangent lines in $T(x, \pi_2)$, are pairwise different. Using this projection we can also determine the tangent line $T_y([y, z])$ (this is the unique line in $T(x, \pi_2)$ mapped to the line containing the projections of y and z). Corollary 4.8 now allows us to determine all tangent lines through y, in particular L.

By Corollary 3.6 the points in $X \cap (\bigcup_{j=1}^{d+1} \pi_j)$ uniquely determine ξ_1 and analogously ξ_2. The point $x \in \xi_1 \cap \xi_2$ is hence also uniquely determined. $\quad\square$

This concludes the proof of the Main Result. As pointed out by the referee, an alternative approach could be to generalize Morin's result on families of intersecting subspaces [11].

Acknowledgements

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