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A localized nonstandard stabilizer for the Timoshenko beam

Un stabilisateur localisé non standard pour la poutre de Timoshenko

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A R T I C L E   I N F O

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A B S T R A C T

The stabilization of the Timoshenko beam system with localized damping is examined. The damping involves the sum of the bending and shear angle velocities; this work generalizes an earlier result of Haraux, established for a system of ordinary wave equations, to the Timoshenko system. First, we show that strong stability holds if and only if the boundary of the support of the feedback control intersects that of the interval under consideration. Next, we use the frequency domain method combined with the multipliers technique to prove the exponential stability of the associated semigroup when the damping support is a neighborhood of one endpoint of the interval under consideration. When the speed of propagation of the wave generated by the bending and that of the wave generated by the shear angle are distinct, the proof is similar to what is known for two ordinary waves similarly damped. However, when the two speeds are equal, an important identity breaks down, and the proof is carried out by the introduction of an appropriate auxiliary equation whose solution plays a critical role in subsequent estimates.

R É S U M É

Dans cette note est examinée la stabilisation de la poutre de Timoshenko avec un amortissement localisé. L'amortissement est lié à la somme des vitesses de tassement et de cisaillement angulaire ; ce travail généralise au système de Timoshenko un résultat antérieur de Haraux, établi pour un système d'équations d'ondes ordinaires. D'abord, nous montrons que la stabilité forte a lieu si et seulement si le support du contrôle rencontre une extrémité de l'intervalle considéré. Puis nous utilisons la combinaison de la méthode des multiplicateurs avec la méthode du domaine des fréquences pour démontrer la stabilité exponentielle du semi-groupe associé quand le support du contrôle rencontre une extrémité de l'intervalle considéré. Quand la vitesse de propagation de l'onde générée par le tassement et celle de l'onde générée par l'angle de cisaillement sont distinctes, la preuve est semblable à celle connue pour deux ondes amorties de la même manière. Cependant, quand les deux vitesses sont égales, une identité importante perd sa validité, et la preuve se poursuit par l'introduction d'une équation auxiliaire dont la solution joue un rôle prépondérant dans les estimations ultérieures.

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1. Introduction and statement of the main results

The stabilization of the Timoshenko beam with various damping mechanisms has been the subject of extensive research over the years. In particular, it is known that when a single dissipation acting through the shear angle equation is utilized, the associated semigroup is exponentially stable if and only if the speeds of propagation of the corresponding waves are equal, e.g. [1,2,4,7,13,15,16,18]:

$$\frac{k}{\rho_1}=\frac{\sigma}{\rho_2}. \tag{1}$$

Condition (1), though mathematically reasonable, is physically unrealistic; it is never satisfied in the real world [12, p. 56]. This last observation may have led some authors to use two feedback controls to exponentially stabilize the Timoshenko system independently of (1); e.g. [3,6,11]. The present note fits in the latter framework, but the damping scheme involves the same feedback control in both equations unlike the cited works where two independent controls are employed. This contribution is motivated by an earlier work of Haraux involving a system of wave equations with different speeds of propagation [8,9].

For the sequel, we need some notations. Let $\Omega=(0,L)$, where L is a positive real number. Let $\omega=(l_1,l_2)$ with $0\le l_1<l_2\le L$, and let $a\in L^\infty(\Omega)$, be a nonnegative function satisfying:

$$\exists a_0>0: a(x)\ge a_0,\quad \text{a.e. } x\in\omega. \tag{2}$$

Throughout the note, we denote by $|u|_r$ the norm of a function $u\in L^r(\Omega)$, $1\le r\le\infty$.

Consider the following damped Timoshenko system

$$\begin{cases}\rho_1 y_{tt}-k(y_x+z)_x+a(y_t+z_t)=0 & \text{in } (0,L)\times(0,\infty)\\ \rho_2 z_{tt}-\sigma z_{xx}+k(y_x+z)+a(y_t+z_t)=0 & \text{in } (0,L)\times(0,\infty),\end{cases} \tag{3}$$

with the boundary conditions (DD stands for Dirichlet–Dirichlet, while DN represents Dirichlet–Neuman):

(**DD**) $y(0,t)=0,\quad y(L,t)=0,\quad z(0,t)=0,\quad z(L,t)=0,$ or else
(**DN**) $y(0,t)=0,\quad y(L,t)=0,\quad z_x(0,t)=0,\quad z_x(L,t)=0,\quad t>0$

and the initial conditions: $y(x,0)=y^0(x)$, $y_t(x,0)=y^1(x)$, $z(x,0)=z^0(x)$, $z_t(x,0)=z^1(x)$, $x\in\Omega$. System (3) describes the motion of a beam when the effects of rotatory inertia are accounted for; the transverse displacement is represented by y while z denotes the shear angle displacement. The constants ρ_1, ρ_2, k, σ are physical constants and are all positive. Condition (2) ensures that the damping is effective in ω. It is worthwhile noting that the damping that is being used is somehow degenerate since the matrix defining it is singular; this makes the stabilization problem more challenging, and worth investigating.

Introduce the energy

$$E(t)=\frac{1}{2}\int_\Omega\left\{\rho_1\left|y_t(x,t)\right|^2+k\left|y_x(x,t)+z(x,t)\right|^2+\rho_2\left|z_t(x,t)\right|^2+\sigma\left|z_x(x,t)\right|^2\right\}\mathrm{d}x,\quad \forall t\ge 0. \tag{4}$$

The energy E is a nonincreasing function of the time variable t as we have for every $t\ge 0$ (hereafter, $'$ denotes differentiation with respect to time)

$$E'(t)=-\int_\Omega a(x)\left|y_t(x,t)+z_t(x,t)\right|^2\mathrm{d}x. \tag{5}$$

Our main purpose in this note is to answer the following questions:

- does the energy $E(t)$ decay to zero as the time variable t goes to infinity?
- if so, how fast? And if not, why?

To study the stabilization problem at hand, we are going to recast System (3) as an abstract evolution system. To this end, setting $Z=\begin{pmatrix}y\\y'\\z\\z'\end{pmatrix}$, (3) may then be recast as: $Z'-\mathcal{A}Z=0$ in $(0,\infty)$, $Z(0)=\begin{pmatrix}y^0\\y^1\\z^0\\z^1\end{pmatrix}$ where the unbounded operator $\mathcal{A}$ is given by

$$\mathcal{A}=\begin{pmatrix}0 & I & 0 & 0\\ \frac{k}{\rho_1}\partial_x^2 & -\frac{a}{\rho_1}I & \frac{k}{\rho_1}\partial_x & -\frac{a}{\rho_1}I\\ 0 & 0 & 0 & I\\ -\frac{k}{\rho_2}\partial_x & -\frac{a}{\rho_2}I & \frac{\sigma}{\rho_2}\partial_x^2-\frac{k}{\rho_2}I & -\frac{a}{\rho_2}I\end{pmatrix} \tag{6}$$

with, in the (**DD**) case: $D(\mathcal{A})=\{(u,v,w,z)\in(H_0^1(\Omega)\times H_0^1(\Omega))^2; k(u_x+w)_x-a(v+z)\in L^2(\Omega),$ and $\sigma w_{xx}-k(u_x+w)-a(v+z)\in L^2(\Omega)\}$ and, in the (**DN**) case: $D(\mathcal{A})=\{(u,v,w,z)\in(H_0^1(\Omega))^2\times V^2; k(u_x+w)_x-a(v+z)\in L^2(\Omega),$ and $\sigma w_{xx}-k(u_x+w)-a(v+z)\in L^2(\Omega)\}$ where $V=\{u\in H^1(\Omega); \int_\Omega u(x)\,\mathrm{d}x=0\}$.

It can easily be checked that in the (**DD**) case, one has $D(\mathcal{A}) = ((H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega))^2$, while in the (**DN**) case, $D(\mathcal{A}) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \times (H^2(\Omega) \cap V) \times V$. Thus, in either case, the operator $\mathcal{A}$ has a compact resolvent. Consequently, the spectrum of $\mathcal{A}$ is discrete.

Introduce the Hilbert spaces over the field $\mathbb{C}$ of complex numbers $\mathcal{H}_1 = (H_0^1(\Omega) \times L^2(\Omega))^2$ and $\mathcal{H}_2 = H_0^1(\Omega) \times L^2(\Omega) \times V \times L^2(\Omega)$, equipped with the norm

$$\|Z\|^2_{\mathcal{H}_i} = \int_\Omega \left\{\rho_1|v|^2 + k|u_x + w|^2 + \rho_2|z|^2 + \sigma|w_x|^2\right\} \mathrm{d}x, \quad \forall Z = (u, v, w, z) \in \mathcal{H}_i,\ i = 1, 2. \tag{7}$$

Our main results read:

Theorem 1.1. *Suppose that ω is an arbitrary nonempty open set in Ω. Let the damping coefficient a be positive in ω. In either of the* (**DD**) *or* (**DN**) *case, the associated operator $\mathcal{A}$ generates a C_0 semigroup of contractions $(S_i(t))_{t\ge 0}$ on the corresponding Hilbert space $\mathcal{H}_i$ ($i = 1, 2$), which is strongly stable:*

$$\lim_{t\to\infty} \left\|S_i(t)Z^0\right\|_{\mathcal{H}_i} = 0, \quad \forall Z^0 \in \mathcal{H}_i, \tag{8}$$

if and only if $\partial\omega \cap \partial\Omega \neq \emptyset$.

Theorem 1.2. *Suppose that ω is an arbitrary nonempty open set in Ω with $\partial\omega \cap \partial\Omega \neq \emptyset$. Let the damping coefficient a satisfy (2). For each $i = 1, 2$, the semigroup $(S_i(t))_{t\ge 0}$ is exponentially stable, viz., there exist positive constants M and λ with*

$$\left\|S_i(t)Z^0\right\|_{\mathcal{H}_i} \le M \exp(-\lambda t)\left\|Z^0\right\|_{\mathcal{H}_i}, \quad \forall Z^0 \in \mathcal{H}_i. \tag{9}$$

Remark 1.3. The part of Theorem 1.1 about strong stability is quite surprising since in order for strong stability to hold, it is necessary that ω be equal to the whole interval Ω or that its closure contain one of the endpoints of Ω. This is the first time that I notice such a restriction in a one-dimensional stabilization problem involving wave equations; usually stabilization results for one-dimensional wave equations hold for every nonempty subset ω. Nevertheless, the structure of the Timoshenko system and that of the feedback control being used are the features imposing such a constraint. The situation here is completely different from what occurs in the case of the system of wave equations addressed in [8,9,17], where strong (and even exponential) stability holds for every nonempty subset ω in the one-dimensional setting, provided the speeds of propagation are pairwise distinct. As we shall see below, when the nonempty intersection condition fails, the operator $\mathcal{A}$ possesses imaginary eigenvalues, and so (3) has solutions with constant energy.

Remark 1.4. As far as Theorem 1.2 is concerned, exponential decay of the energy has been established in the literature when:

- the matrix defining the damping is positive definite; this is equivalent to using two independent dampings, one in the bending equation and one in the shear angle equation,
- one damping is used in the shear angle equation only.

In either of those cases, exponential decay of the energy is established without the nonempty intersection constraint, but in the latter case with the assumption of equal speeds of propagation (1), to compensate for the use of a single feedback control. In the former case, the fact that the damping matrix is positive definite makes the corresponding stabilization problem much simpler than the one that we are dealing with in this note.

To the best of my knowledge, all other cases, including the case at hand, are open. For the sequel, we will be using the following additional notations: $\hat{k} = k/\rho_1$, $\check{k} = k/\rho_2$, $\hat{a} = a/\rho_1$, $\check{a} = a/\rho_2$, $\hat{\sigma} = \sigma/\rho_2$.

2. Ideas for proving Theorem 1.1

The semigroup generation is pretty straightforward thanks to Lumer–Phillips Theorem. We will focus our attention on strong stability. Since in either case the operator $\mathcal{A}$ has a compact resolvent, its spectrum is discrete. Thus, to prove strong stability, it suffices, thanks to a result in [5], to show that $\mathcal{A}$ has no imaginary eigenvalue. One easily checks that zero is not an eigenvalue of $\mathcal{A}$. Now, let b be a nonzero real number, and let $Z = (u, v, w, z) \in D(\mathcal{A})$ such that $\mathcal{A}Z = ibZ$. We shall prove that $Z = (0, 0, 0, 0)$ if and only if $\partial\omega \cap \partial\Omega \neq \emptyset$. Note that $\mathcal{A}Z = ibZ$ may be reduced to:

$$\begin{cases} -b^2 u - \hat{k}(u_x + w)_x + ib\hat{a}(u + w) = 0 & \text{in } (0, L) \\ -b^2 w - \hat{\sigma} w_{xx} + \check{k}(u_x + w) + ib\check{a}(u + w) = 0 & \text{in } (0, L). \end{cases} \tag{10}$$

One easily checks that $u = -w$ in ω, by multiplying the first equation by $\bar{u}$, the second by $\bar{w}$, integrating by parts and taking the imaginary parts. The heart of the matter now is to check under which condition, one has $u \equiv 0$ in ω; indeed, if we can prove that $u \equiv 0$ in ω, then $w \equiv 0$ in ω, and basic uniqueness results may then be invoked to conclude that $Z = (0, 0, 0, 0)$. Therefore, it remains to find under which condition $u \equiv 0$ in ω. Since $u = -w$ in ω, System (10) becomes:

$$\begin{cases} -b^2 u - \hat{k}(u_x - u)_x = 0 & \text{in } \omega \\ b^2 u + \hat{\sigma} u_{xx} + \check{k}(u_x - u) = 0 & \text{in } \omega. \end{cases} \tag{11}$$

Adding both equations in (11), one gets rid of b, thereby obtaining

$$(\hat{\sigma}-\hat{k})u_{xx}+(\hat{k}+\check{k})u_x-\check{k}u=0 \quad \text{in } \omega. \tag{12}$$

The characteristic equation for (12) is given by

$$(\hat{\sigma}-\hat{k})r^2+(\hat{k}+\check{k})r-\check{k}=0. \tag{13}$$

So, if $\hat{\sigma}=\hat{k}$, then one gets

$$r=\check{k}/(\hat{k}+\check{k}), \qquad u(x)=A\,\mathrm{e}^{rx} \quad \text{for every } x\in\omega, \tag{14}$$

where A is an arbitrary constant.

Consequently, if $\partial\omega\cap\partial\Omega\neq\emptyset$, then automatically $A=0$, (no matter which homogeneous boundary conditions u may satisfy), and so $u\equiv 0$ in ω. On the other hand, $\pm\mathrm{i}\frac{\hat{k}\sqrt{\check{k}}}{\hat{k}+\check{k}}$ are eigenvalues of $\mathcal{A}$, in either case, for $\partial\omega\cap\partial\Omega=\emptyset$.

If $\hat{\sigma}\neq\hat{k}$, then the solutions of (13) and (12) are given by

$$r_\pm=\frac{-(\hat{k}+\check{k})\pm\sqrt{(\hat{k}-\check{k})^2+4\hat{\sigma}\check{k}}}{2(\hat{\sigma}-\hat{k})}, \qquad u(x)=A\,\mathrm{e}^{r_+x}+B\,\mathrm{e}^{r_-x}, \quad \text{for every } x\in\omega, \tag{15}$$

where A and B are arbitrary constants.

One checks that the constants A and B satisfy the equations

$$\begin{aligned}&\left(-b^2-\hat{k}r_+^2+\hat{k}r_+\right)A=0, \qquad \left(-b^2-\hat{k}r_-^2+\hat{k}r_-\right)B=0,\\ &\left(b^2+\hat{\sigma}r_+^2+\check{k}r_+-\check{k}\right)A=0, \qquad \left(b^2+\hat{\sigma}r_-^2+\check{k}r_--\check{k}\right)B=0.\end{aligned} \tag{16}$$

First assume $\partial\omega\cap\partial\Omega\neq\emptyset$, then $A=-B\,\mathrm{e}^{(r_--r_+)x_0}$ with $x_0=0$ or else $x_0=L$; therefore $u(x)=B(-\mathrm{e}^{(r_--r_+)x_0+r_+x}+\mathrm{e}^{r_-x})$ in ω. For $\hat{\sigma}>\hat{k}$, one has $r_-<0$ so that by the top right equation in (16), one derives $B=0$; hence $u\equiv 0$ in ω. For $\hat{\sigma}<\hat{k}$, one derives $r_->1$ so that by the bottom right equation in (16), it follows once more $B=0$; whence $u\equiv 0$ in ω. Next, we turn to the case $\partial\omega\cap\partial\Omega=\emptyset$; here, we note $\pm\mathrm{i}\sqrt{\hat{k}r_+(1-r_+)}$ are eigenvalues of $\mathcal{A}$, (note that $r_+<1$ whatever the sign of $\hat{\sigma}-\hat{k}$ might be). In short, we have just shown that the operator $\mathcal{A}$ has no imaginary eigenvalue as long as $\partial\omega\cap\partial\Omega\neq\emptyset$, while it does have imaginary eigenvalues when $\partial\omega\cap\partial\Omega=\emptyset$. Hence the associated semigroup is strongly stable if and only if $\partial\omega\cap\partial\Omega\neq\emptyset$. □

3. Ideas for proving Theorem 1.2

By Theorem 1.1, we already know that strong stability holds only when $\partial\omega\cap\partial\Omega\neq\emptyset$. So this condition is necessary for exponential decay of the semigroup too. We are going to show that it is also sufficient. To this end and for the sake of clarity, we set $\omega=(l_1,L)$, and limit ourselves to the case (**DD**); the case (**DN**) being similar. For simplicity sake, we use the notation $\mathcal{H}=\mathcal{H}_1$. We shall use the frequency domain approach, which amounts to showing the two facts [10,14]:

i) $\mathrm{i}\mathbb{R}\subset\rho(\mathcal{A})$, and
ii) $\sup\{\|(\mathrm{i}b-\mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})};\ b\in\mathbb{R}\}<\infty$, where $\rho(\mathcal{A})$ denotes the resolvent set of $\mathcal{A}$.

Thanks to the proof of Theorem 1.1, we already have i). It remains to prove ii). For this purpose, it suffices to show that there exists $C_0>0$ such that for every $U\in\mathcal{H}$, one has:

$$\left\|(\mathrm{i}b-\mathcal{A})^{-1}U\right\|_{\mathcal{H}}\le C_0\|U\|_{\mathcal{H}}, \quad \forall b\in\mathbb{R}, \tag{17}$$

where hereafter, C_0 denotes a generic positive constant that may eventually depend on Ω, ω, and the other parameters of the system, but never on b. Let $b\in\mathbb{R}$, $U=(f,g,h,l)\in\mathcal{H}$, and let $Z=(u,v,w,z)\in D(\mathcal{A})$ such that

$$(\mathrm{i}b-\mathcal{A})Z=U. \tag{18}$$

We shall prove $\|Z\|_{\mathcal{H}}\le C_0\|U\|_{\mathcal{H}}$. To this end, multiply both sides of (18) by Z, then take the real part of the inner product in $\mathcal{H}$ to derive:

$$\int_\Omega a(x)\left|v(x)+z(x)\right|^2\mathrm{d}x=\Re(U,Z)\le\|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}}. \tag{19}$$

Eq. (18) may be recast as:

$$\begin{cases}\mathrm{i}bu-v=f\\ \mathrm{i}bv-\hat{k}(u_x+w)_x+\hat{a}(v+z)=g\\ \mathrm{i}bw-z=h\\ \mathrm{i}bz-\hat{\sigma}w_{xx}+\check{k}(u_x+w)+\check{a}(v+z)=l.\end{cases} \tag{20}$$

Thanks to (19), we derive

$$b^2 \int_\Omega a(x)\big|u(x)+w(x)\big|^2\,\mathrm{d}x \le 2\|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}} + C_0\|U\|_{\mathcal{H}}^2. \tag{21}$$

For the sequel, we now introduce a cut-off function that will be used to build needed multipliers. Let $q \in C^2([0,L])$ such that $q \equiv 1$ on $[0, l_1+\varepsilon]$, $q \equiv 0$ on $[l_1+2\varepsilon, L]$, for some $\varepsilon > 0$ with $l_1 + 2\varepsilon < L$, and set $\omega_\varepsilon = (l_1+\varepsilon, L)$.

Multiplying the second equation in (20) by $2xq(\bar{u}_x+\bar{w})$, using the first equation in (20), taking the real part, and integrating by parts over Ω, one finds

$$\begin{aligned} b^2|u|_2^2 + \hat{k}|u_x+w|_2^2 = {} & \int_\Omega (1-q-xq_x)\big(b^2|u|^2+\hat{k}|u_x+w|^2\big)\,\mathrm{d}x + 2\Re\int_\Omega \big(g-\hat{a}(v+z)\big)xq(\bar{u}_x+\bar{w})\,\mathrm{d}x \\ & - 2\Re\int_\Omega \mathrm{i}b\bar{u}(xqf)_x\,\mathrm{d}x - 2\Re\int_\Omega \mathrm{i}bv\bar{w}xq\,\mathrm{d}x. \end{aligned} \tag{22}$$

Thanks to the Cauchy–Schwarz and Poincaré inequalities, the first and the third equations in (20) as well as (19), it follows from (22):

$$\begin{aligned} b^2|u|_2^2 + \hat{k}|u_x+w|_2^2 \le {} & C_0\int_{\omega_\varepsilon} \big(b^2|u|^2+|u_x+w|^2\big)\,\mathrm{d}x + C_0\big(|g|_2|u_x+w|_2 + \big|\sqrt{a}(v+z)\big|_2|u_x+w|_2 \\ & + |v|_2|f_x|_2 + |f_x|_2^2 + |v|_2|h_x|_2\big) + 2\Re\int_\Omega v\bar{z}xq\,\mathrm{d}x \\ \le {} & C_0\int_{\omega_\varepsilon}\big(b^2|u|^2+|u_x+w|^2\big)\,\mathrm{d}x + C_0\big(\|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{\frac{1}{2}}\|Z\|_{\mathcal{H}}^{\frac{3}{2}} + \|U\|_{\mathcal{H}}^2\big) + 2\Re\int_\Omega v\bar{z}xq\,\mathrm{d}x. \end{aligned} \tag{23}$$

Multiplying the fourth equation in (20) by $2xq(\hat{\sigma}\bar{w}_x - \check{k}\bar{u})$, using the third equation in (20), taking the real part, and integrating by parts over Ω, one obtains:

$$\begin{aligned} \hat{\sigma}b^2|w|_2^2 + \hat{\sigma}^2|w_x|_2^2 = {} & \hat{\sigma}\int_\Omega (1-q-xq_x)\big(b^2|w|^2+\hat{\sigma}|w_x|^2\big)\,\mathrm{d}x + 2\Re\int_\Omega \big(l-\check{a}(v+z)\big)xq(\hat{\sigma}\bar{w}_x-\check{k}\bar{u})\,\mathrm{d}x \\ & - 2\hat{\sigma}\Re\int_\Omega \mathrm{i}b\bar{w}(xqh)_x\,\mathrm{d}x + 2\check{k}\Re\int_\Omega \mathrm{i}bz\bar{u}xq\,\mathrm{d}x + 2\check{k}^2\Re\int_\Omega (u_x+w)\bar{u}xq\,\mathrm{d}x \\ & - 2\check{k}\hat{\sigma}\Re\int_\Omega \big(\bar{w}_xwxq - w_x\bar{u}(xq)_x\big)\,\mathrm{d}x. \end{aligned} \tag{24}$$

Invoking the Cauchy–Schwarz inequality as well as first and third equations in (20) once more, we derive from (24), and for every nonzero b:

$$\begin{aligned} \hat{\sigma}b^2|w|_2^2 + \hat{\sigma}^2|w_x|_2^2 \le {} & C_0\int_{\omega_\varepsilon}\big(b^2|w|^2+|w_x|^2\big)\,\mathrm{d}x + C_0\big(\|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{\frac{1}{2}}\|Z\|_{\mathcal{H}}^{\frac{3}{2}} + \|U\|_{\mathcal{H}}^2\big) + \frac{b^2}{2}\big(\check{k}|u|_2^2+\hat{\sigma}|w|_2^2\big) \\ & + \frac{C_0}{b^2}\big(|u_x+w|_2^2+|w_x|_2^2\big) - 2\check{k}\Re\int_\Omega z\bar{v}xq\,\mathrm{d}x. \end{aligned} \tag{25}$$

Multiplying (23) by $\check{k}$, adding the result to (25), and choosing b with large enough absolute value, we find:

$$\begin{aligned} \check{k}b^2|u|_2^2 + \check{k}\hat{k}|u_x+w|_2^2 + \hat{\sigma}b^2|w|_2^2 + \hat{\sigma}^2|w_x|_2^2 \le {} & C_0\int_{\omega_\varepsilon}\big(b^2\big(|u|^2+|w|^2\big)+|u_x+w|^2+|w_x|^2\big)\,\mathrm{d}x \\ & + C_0\big(\|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{\frac{1}{2}}\|Z\|_{\mathcal{H}}^{\frac{3}{2}} + \|U\|_{\mathcal{H}}^2\big). \end{aligned} \tag{26}$$

At this stage, we note that it remains to absorb the first term in the right-hand side of (26). First, we are going to absorb the terms involving u_x and w_x. For this purpose, we now introduce another cut-off function; let $\eta \in C^2([0,L])$ with $\eta \equiv 1$

on ω_ε, and $\eta \equiv 0$ on $[0, l_1]$. Multiplying the second equation in (20) by $\eta^2 \bar{u}$, the fourth equation by $\eta^2 \bar{w}$, and proceeding as above, one derives:

$$C_0 \int_{\omega_\varepsilon} \left(|u_x + w|^2 + |w_x|^2\right) \mathrm{d}x \le C_0 b^2 \int_\Omega \eta^2 \left(|u|^2 + |w|^2\right) \mathrm{d}x + \frac{C_0}{b^2}\left(\hat{k}\check{k}|u_x + w|_2^2 + \hat{\sigma}^2 |w_x|_2^2\right) + \frac{b^2}{2}\left(\check{k}|u|_2^2 + \hat{\sigma}|w|_2^2\right) + C_0\left(\|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{\frac{1}{2}}\|Z\|_{\mathcal{H}}^{\frac{3}{2}} + \|U\|_{\mathcal{H}}^2\right), \tag{27}$$

where C_0 in the left hand side is the same as in (26). Choosing b with large enough $|b|$, and combining (26) and (27), we find:

$$\begin{aligned}
&\check{k} b^2 |u|_2^2 + \check{k}\hat{k}|u_x + w|_2^2 + \hat{\sigma} b^2 |w|_2^2 + \hat{\sigma}^2 |w_x|_2^2 \\
&\quad \le C_0 b^2 \int_\Omega \eta^2 \left(|u|^2 + |w|^2\right) \mathrm{d}x + C_0\left(\|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{\frac{1}{2}}\|Z\|_{\mathcal{H}}^{\frac{3}{2}} + \|U\|_{\mathcal{H}}^2\right) \\
&\quad \le C_0 b^2 \int_\Omega \eta^2 |u + w|^2 \,\mathrm{d}x - 2C_0 b^2 \Re \int_\Omega \eta^2 u \bar{w} \,\mathrm{d}x + C_0\left(\|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{\frac{1}{2}}\|Z\|_{\mathcal{H}}^{\frac{3}{2}} + \|U\|_{\mathcal{H}}^2\right).
\end{aligned} \tag{28}$$

Now, it remains to dispose of the term involving the product $u\bar{w}$. To this end, multiply the second equation in (20) by $\hat{\sigma}\eta^2 \bar{w}$ and the fourth equation by $\hat{k}\eta^2 \bar{u}$, integrate by parts over Ω, then subtract the fourth equation from the second to get, (setting $\hat{g} = g - \hat{a}(v + z)$ and $\hat{l} = l - \check{a}(v + z)$)

$$\begin{aligned}
-b^2(\hat{\sigma} - \hat{k})\Re \int_\Omega \eta^2 u \bar{w} \,\mathrm{d}x &= \hat{\sigma} \Re \int_\Omega \left((\hat{g} + \mathrm{i}bf)\eta^2 - 2\hat{k}\eta_x \eta (u_x + w)\right)\bar{w} \,\mathrm{d}x \\
&\quad - \hat{k}\Re \int_\Omega \left(\left(\hat{l} + \mathrm{i}bh - \check{k}(u_x + w)\right)\eta^2 \bar{u} - 2\hat{\sigma}\eta_x \eta w_x \bar{u} + \hat{\sigma}\bar{w}_x w\right) \mathrm{d}x.
\end{aligned} \tag{29}$$

At this stage, we note that when $\hat{k} \ne \hat{\sigma}$, one can easily estimate the left-hand side of (29) as above, so that reporting the resulting estimate in (28), one finds:

$$\check{k} b^2 |u|_2^2 + \check{k}\hat{k}|u_x + w|_2^2 + \hat{\sigma} b^2 |w|_2^2 + \hat{\sigma}^2 |w_x|_2^2 \le C_0\left(\|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{\frac{1}{2}}\|Z\|_{\mathcal{H}}^{\frac{3}{2}} + \|U\|_{\mathcal{H}}^2\right), \tag{30}$$

from which one easily derives (17) for all b with $|b| > b_0$ for some large enough b_0. For $|b| \le b_0$, one invokes the continuity of the resolvent. This completes the proof in the case of different wave speeds. We now turn to the case of equal wave speeds: $\hat{k} = \hat{\sigma}$, which is the most interesting one because the identity (29), which is critical in the case of different wave speeds, breaks down (we are no longer able to estimate the left-hand side of (29)), and another approach is needed. For this purpose, we note that multiplying the second equation in (20) by $\eta^2 \bar{w}$, and integrating by parts over Ω, one obtains, after some algebra

$$-b^2 \Re \int_\Omega \eta^2 u \bar{w} \,\mathrm{d}x = -\hat{k}\Re \int_\Omega \eta^2 u_x \bar{w}_x \,\mathrm{d}x - \hat{k}\Re \int_\Omega \left(\eta^2 w_x + 2\eta_x \eta (u_x + w)\right)\bar{w} \,\mathrm{d}x + \Re \int_\Omega (\hat{g} + \mathrm{i}bf)\eta^2 \bar{w} \,\mathrm{d}x. \tag{31}$$

Here, we draw the reader's attention to the fact that a careful examination of (31) shows that estimating the term involving the product $u\bar{w}$ in (28) amounts to estimating the term involving the product $u_x \bar{w}_x$ in (31). Now, set $\varphi = u + w$, then $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$ and satisfies the equation (keep in mind that $\hat{k} = \hat{\sigma}$)

$$-b^2 \varphi - \hat{k}\varphi_{xx} = \hat{g} + \hat{l} + \mathrm{i}b(f + h) + \hat{k}w_x - \check{k}(u_x + w). \tag{32}$$

Multiplying (32) by $\eta^2 \bar{\varphi}$, integrating by parts over Ω, and using (21) as well as the Cauchy–Schwarz and Poincaré inequalities, one finds:

$$\begin{aligned}
\int_\Omega \eta^2 |\varphi_x|^2 \,\mathrm{d}x &= b^2 \int_\Omega \eta^2 |\varphi|^2 \,\mathrm{d}x + \Re \int_\Omega \left\{\hat{g} + \hat{l} + \mathrm{i}b(f + h) + \hat{k}w_x - \check{k}(u_x + w)\right\}\eta^2 \bar{\varphi} \,\mathrm{d}x \\
&\le b^2 |\varphi|_2^2 + C_0\left(|g + l|_2^2 + |\sqrt{a}(v + z)|_2^2 + b^2 |\varphi|_2^2 + |f_x + h_x|_2^2 + \left(|w_x|_2 + |u_x + w|_2\right)|\varphi|_2\right) \\
&\le C_0\left(\|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^2 + \|U\|_{\mathcal{H}}^{\frac{1}{2}}\|Z\|_{\mathcal{H}}^{\frac{3}{2}}\right).
\end{aligned} \tag{33}$$

Noting that $\hat{k}|\varphi_x(L)|^2 = \hat{k}\eta^4(L)|\varphi_x(L)|^2 = 2\hat{k}\Re \int_\Omega \eta^2 \bar{\varphi}_x(\eta^2 \varphi_{xx} + 2\eta_x \eta \varphi_x) \,\mathrm{d}x$, and using (32), integration by parts, the Cauchy–Schwarz inequality as well as (21) and (33), we derive:

$$\left|\varphi_x(L)\right|^2 \le C_0\left(\|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{\frac{1}{2}}\|Z\|_{\mathcal{H}}^{\frac{3}{2}} + \|U\|_{\mathcal{H}}^2 + \|U\|_{\mathcal{H}}^{\frac{1}{4}}\|Z\|_{\mathcal{H}}^{\frac{7}{4}}\right). \tag{34}$$

Similarly, one checks that

$$\left|w_x(L)\right|^2 \le C_0\left(\|U\|_{\mathcal{H}}^2 + \|Z\|_{\mathcal{H}}^2\right). \tag{35}$$

Multiplying (32) by $\eta^2\bar{w}_x$, integrating by parts over Ω, and using the Cauchy–Schwarz inequality as well as (21) and (33)–(35) yield:

$$\begin{aligned}
&\hat{k}\int_\Omega \eta^2|w_x|^2\,dx - \check{k}\Re\int_\Omega \eta^2 u_x\bar{w}_x\,dx = -\Re\int_\Omega \big(\eta^2 h_x + 2\eta_x\eta(ib\bar{w}+\bar{h})\big)ib\varphi\,dx - \hat{k}\Re\varphi_x(L)\overline{w_x(L)}\\
&\quad + \Re\int_\Omega \eta^2\varphi_x\big(\check{k}(\bar{u}_x+\bar{w})-\bar{\hat{l}}\big)\,dx - \Re\int_\Omega \eta^2(\hat{g}+\hat{l}-\check{k}w)\bar{w}_x - ib\big(\eta^2(f+h)\big)_x\bar{w}\,dx\\
&\quad \le C_0\big(\|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{\frac12}\|Z\|_{\mathcal{H}}^{\frac32} + \|U\|_{\mathcal{H}}^2 + \|U\|_{\mathcal{H}}^{\frac14}\|Z\|_{\mathcal{H}}^{\frac74} + \|U\|_{\mathcal{H}}^{\frac18}\|Z\|_{\mathcal{H}}^{\frac{15}{8}}\big) + C_0|w|_2^2.
\end{aligned}\tag{36}$$

On the other hand, thanks to (33), one has the following estimate

$$\int_\Omega \eta^2|w_x|^2\,dx + \Re\int_\Omega \eta^2 u_x\bar{w}_x\,dx = \Re\int_\Omega \eta^2\varphi_x\bar{w}_x\,dx \le C_0\big(\|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{\frac12}\|Z\|_{\mathcal{H}}^{\frac32} + \|U\|_{\mathcal{H}}^{\frac14}\|Z\|_{\mathcal{H}}^{\frac74}\big).\tag{37}$$

It now follows from (36) and (37)

$$\left|\Re\int_\Omega \eta^2 u_x\bar{w}_x\,dx\right| \le C_0\big(\|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{\frac12}\|Z\|_{\mathcal{H}}^{\frac32} + \|U\|_{\mathcal{H}}^2 + \|U\|_{\mathcal{H}}^{\frac14}\|Z\|_{\mathcal{H}}^{\frac74} + \|U\|_{\mathcal{H}}^{\frac18}\|Z\|_{\mathcal{H}}^{\frac{15}{8}}\big) + C_0|w|_2^2.\tag{38}$$

Thanks to the Cauchy–Schwarz and Young inequalities, one derives from (31):

$$\begin{aligned}
b^2\left|\Re\int_\Omega \eta^2 u\bar{w}\,dx\right| &\le \hat{k}\left|\Re\int_\Omega \eta^2 u_x\bar{w}_x\,dx\right| + \frac{C_0}{b^2}\big(|u_x+w|_2^2 + |w_x|_2^2\big) + \frac{\hat{\sigma}b^2}{2}|w|_2^2\\
&\quad + C_0\big(\|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{\frac12}\|Z\|_{\mathcal{H}}^{\frac32} + \|U\|_{\mathcal{H}}^2\big).
\end{aligned}\tag{39}$$

Combining (38)–(39), reporting the result in (28), we get for large enough $|b|$:

$$\begin{aligned}
&\check{k}b^2|u|_2^2 + \check{k}\hat{k}|u_x+w|_2^2 + \hat{\sigma}b^2|w|_2^2 + \hat{\sigma}^2|w_x|_2^2\\
&\quad \le C_0\big(\|U\|_{\mathcal{H}}\|Z\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{\frac12}\|Z\|_{\mathcal{H}}^{\frac32} + \|U\|_{\mathcal{H}}^2 + \|U\|_{\mathcal{H}}^{\frac14}\|Z\|_{\mathcal{H}}^{\frac74} + \|U\|_{\mathcal{H}}^{\frac18}\|Z\|_{\mathcal{H}}^{\frac{15}{8}}\big),
\end{aligned}\tag{40}$$

from which one derives (17) for all b with $|b| > b_1$ for some large enough b_1. For $|b| \le b_1$, one invokes the continuity of the resolvent. This completes the proof in the case of equal wave speeds and that of the theorem. □

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